BA'ZI PARAMETRGA BOG'LIQ INTEGRALLARNI HISOBLASH

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ANNOTATSIYA

Ushbu maqolada parametrga bog'liq integrallarning xossalaridan foydalanib, ba`zi parametrga bog'liq integrallarni hisoblash o`rganilgan.

Kalit so‘zlar: parametr, uzluksizlik, differensiallash, Aniq integral, tekis yaqinlashuvchilik.

ABSTRACT

In this article, using the properties of parameter-dependent integrals, the calculation of some parameter-dependent integrals is studied.

Keywords: parameter, continuity, differentiation, Exact integral, smooth convergence.

Bizga $f(x,y)$ funksiya R^2 fazodagi biror

$$M=\{(x,y)\in R^2 : a\le x\le b, c\le y\le d\}$$

to’plamda berilgan bo’lsin. y o’zgaruvchining $[c,d]$ to’plamdan olingan har bir tayinlangan qiymatida $f(x,y)$ funksiya x o’zgaruvchisi bo’yicha $[a,b]$ oraliqda integrallanuvchi, ya’ni

$$\int_a^b f(x,y)dx$$

integral mavjud bo’lsin. Bu integral y o’zgaruvchining $[c,d]$ to’plamdan olingan qiymatiga bog’liq bo’ladi:

$$I(y)=\int_a^b f(x,y)dx \qquad (1)$$

Odatda (1) integral parametrga bog’liq integral deb ataladi, y o’zgaruvchi esa parametr deyiladi.

1-teorema. Agar $f(x,y)$ funksiya M to‘plamda uzluksiz bo‘lsa,

$$I(y)=\int_a^b f(x,y)dx$$

integral mavjud bo‘ladi.

2-teorema. Agar $f(x,y)$ funksiya M to‘plamda uzluksiz bo‘lsa, $I(y)=\int_a^b f(x,y)dx$ integral y bo‘yicha uzluksiz bo‘ladi.

3-teorema (Leybnits qoidasi).

1) $f(x,y)$ funksiya M to‘plamda aniqlangan va uzluksiz;
2) M da $f_y'(x,y)$ mavjud va u ham uzluksiz.

U holda $[c,d]$ da uzluksiz

$$I'(y)=\int_a^b f_y'(x,y)dx$$

mavjud.

4-teorema (Parametr bo‘yicha integrallash).

$M=\{(x,y)\in\mathbb{R}^2 : a\le x\le b,\ c\le y\le d\}$ to‘plamda uzluksiz $f(x,y)$ funksiya uchun $\int_c^d dy\int_a^b f(x,y)dx=\int_c^d I(y)dx=\int_a^b dx\int_c^d f(x,y)dy$ tenglik o‘rinli.

5-teorema.

1) $f(x,y)$ funksiya $M_1=\{(x,y)\in\mathbb{R}^2 : a\le x\le b, c\le y\le d\}$ to‘plamda aniqlangan va uzluksiz;
2) $f_y'(x,y)$ hosila M_1 to‘plamda uzluksiz;
3) $I(y)=\int_a^b f(x,y)dx$ $[c,d]$ da yaqinlashuvchi;
4) $\int_a^b f_y'(x,y)dx$ $[c,d]$ da tekis yaqinlashuvchi.

U holda uzluksiz $I'(y)=\int_a^b f_y'(x,y)dx$ mavjud.

6-teorema. $M=\{(x,y)\in\mathbb{R}^2 : a\le x\le b,\ c\le y\le d\}$ to‘plamda uzluksiz $f(x,y)$ funksiya uchun $F(x)=\int_c^d g(y)f(x,y)dy$ tenglik bilan aniqlangan $F(x)$ funksiya $[a,b]$ kesmada integrallanuvchi bo‘lib,

$$\int_a^b F(x)dx = \int_c^d g(y)dy \int_a^b f(x,y)dx$$

tenglik o'rinli. Bu yerda g funksiyani $[c,d]$ da integrallanuvchi deb faraz qilamiz.

1-misol. $I(\alpha)=\int_0^{\frac{\pi}{2}} \ln(\sin^2 x+\alpha^2\cos^2 x)dx$, $(\alpha>0)$ integralni hisoblang.

Yechish. Ma'lumki,

$$I(1)=\int_0^{\frac{\pi}{2}} \ln(\sin^2 x+\cos^2 x)dx=\int_0^{\frac{\pi}{2}} \ln 1 dx = 0.$$

Bu yerda $f(x,\alpha)=\ln(\sin^2 x+\alpha^2\cos^2 x)$ va $f_\alpha'(x,\alpha)=\dfrac{2\alpha\cos^2 x}{\sin^2 x+\alpha^2\cos^2 x}$

funksiyalar barcha $x\in[0,\frac{\pi}{2}]$ larda va $\alpha>0$ da uzluksiz. U holda Leybnits qoidasiga ko'ra

$$I'(\alpha)=\int_0^{\frac{\pi}{2}} \frac{2\alpha\cos^2 x}{\sin^2 x+\alpha^2\cos^2 x}dx$$

mavjud. $\alpha\neq 1$ da $t=tgx$ almashtirish bajaramiz.

$t=tgx$, $x=arctgt$, $dx=\dfrac{1}{1+t^2}dt$, $0\le x\le\dfrac{\pi}{2}$, $0\le t<+\infty$, u holda

$$I'(\alpha)=2\alpha\int_0^{+\infty}\frac{dt}{(t^2+1)(t^2+\alpha^2)}=\frac{2\alpha}{\alpha^2-1}\int_0^{+\infty}\left(\frac{1}{t^2+1}-\frac{1}{t^2+\alpha^2}\right)dt=$$

$$=\lim_{A\to+\infty}\frac{2\alpha}{\alpha^2-1}\left(arctgt-arctg\frac{t}{\alpha}\right)\Bigg|_0^A=\frac{\pi}{\alpha+1}.$$

Bu yerdan $I(\alpha)=\int\dfrac{\pi}{\alpha+1}d\alpha=\pi\ln(\alpha+1)+C$.

$I(\alpha)$ funksiya $\alpha>0$ da uzluksiz va $I(1)=0$, u holda $I(1)=\pi\ln(1+1)+C$, bu yerdan $C=-\pi\ln 2$ bo'lishi kelib chiqadi. Natijada $I(\alpha)=\pi\left(\ln(\alpha+1)-\ln 2\right)$ bo'lishini topamiz.

2-misol. $I(\alpha)=\int_0^1 \frac{dx}{x^2+\alpha^2}$, $\alpha>0$ integralni parametr bo‘yicha differensiallashdan foydalanib, ushbu $\int_0^1 \frac{dx}{\left(x^2+\alpha^2\right)^2}$ integralni hisoblang.

Yechish. Ma`lumki, Bu yerda integralni parametr bo‘yicha differensiallash va uzluksizligining barcha shartlari bajariladi, chunki $f(x,\alpha)=\frac{1}{x^2+\alpha^2}$ va $f'(x,\alpha)=\frac{-2\alpha}{(x^2+\alpha^2)^2}$ funksiyalar $\alpha>0$ va barcha $x\in[0,1]$ da uzluksiz. U holda

$$I(\alpha)=\int_0^1 \frac{dx}{x^2+\alpha^2}=\frac{1}{\alpha}arctg\frac{x}{\alpha}\Big|_0^1=\frac{arctg\frac{1}{\alpha}}{\alpha}.$$

Endi $I'(\alpha)$ ni topamiz:

$$I'(\alpha)=\left(\frac{arctg\frac{1}{\alpha}}{\alpha}\right)'=-\frac{\alpha+(1+\alpha^2)arctg\frac{1}{\alpha}}{\alpha^2(1+\alpha^2)}.$$

U holda

$$I(\alpha)=\int_0^1 \frac{dx}{\left(x^2+\alpha^2\right)^2}=-\frac{1}{2\alpha}\int_0^1\left(\frac{1}{x^2+\alpha^2}\right)'_\alpha dx=-\frac{I'(\alpha)}{2\alpha}=\frac{\alpha+(1+\alpha^2)arctg\frac{1}{\alpha}}{2\alpha^3(1+\alpha^2)}$$

tenglik o‘rinli bo‘ladi. Demak,

$$I(\alpha)=\frac{\alpha+(1+\alpha^2)arctg\frac{1}{\alpha}}{2\alpha^3(1+\alpha^2)}$$

bo‘ladi.

3-misol. $I=\int_0^1 \frac{x^d-x^c}{\ln x}dx$ integralni hisoblang.

Yechish. $f(x,y)=x^y$ funksiyani qaraymiz. Bu funksiya

$$M_1=\{(x,y)\in\mathbb{R}^2: 0\le x\le 1,\ c\le y\le d\}$$

to‘plamda uzluksiz. 4-teoremaga ko‘ra, ushbu integralda integrallash tartibini (parametr bo‘yicha integrallash) almashtirish mumkin.

$$\int_c^d\left(\int_0^1 x^y dx\right)dy=\int_0^1\left(\int_c^d x^y dy\right)dx=I\,,$$

$$\int_c^d x^y dy=\frac{x^y}{\ln x}\bigg|_c^d=\frac{x^d-x^c}{\ln x}.$$

Lekin, $\int_0^1 x^y dx=\frac{x^{y+1}}{y+1}\bigg|_0^1=\frac{1}{1+y}$. Shuning uchun

$$\int_x^d\left(\int_0^1 x^y dx\right)dy=\int_c^d\frac{1}{1+y}dy=\ln(1+y)|_c^d=\ln\frac{1+d}{1+c}.$$

4-misol. $I(\alpha)=\int_0^1\frac{arctg(\alpha x)}{x\sqrt{1-x^2}}dx$ ni hisoblang.

Yechish. Bu yerda $f(x,\alpha)=\frac{arctg(\alpha x)}{x\sqrt{1-x^2}}$ va $f_\alpha{}'(x,\alpha)=\frac{1}{(1+\alpha^2x^2)\sqrt{1-x^2}}$ lar $M=\{(x,\alpha)\in\mathbb{R}^2:0\le x\le 1,-\infty<\alpha<+\infty\}$ da uzluksiz. Shuningdek,

$$\left|f_\alpha{}'(x,\alpha)\right|=\left|\frac{1}{(1+\alpha^2x^2)\sqrt{1+x}\sqrt{1-x}}\right|\le\frac{1}{\sqrt{1-x}}$$

va $\int_0^1\frac{dx}{\sqrt{1-x}}$ integral yaqinlashuvchi. Chunki,

$$\int_0^1\frac{dx}{\sqrt{1-x}}=\lim_{A\to 1-0}\int_0^A\frac{dx}{\sqrt{1-x}}=-\lim_{A\to 1-0}2(1-x)^{\frac{1}{2}}\bigg|_0^A=-\lim_{A\to 1-0}\left(2(1-A)^{\frac{1}{2}}-2\right)=2$$

U holda Veyershtrass alomatiga ko‘ra $\int_0^1 f_\alpha{}'(x,\alpha)dx$ integralning $(-\infty,\infty)$ da tekis yaqinlashuvchi bo‘lishi kelib chiqadi va 5-teoremaga ko‘ra

$$I'(\alpha)=\int_0^1 f_\alpha{}'(x,\alpha)dx=\int_0^1\frac{dx}{(1+\alpha^2x^2)\sqrt{1-x^2}}$$

bo‘ladi. Oxirgi integralda $x=\sin t$ almashtirish bajaramiz. U holda

$$\int_0^1\frac{dx}{(1+\alpha^2x^2)\sqrt{1-x^2}}=\int_0^{\frac{\pi}{2}}\frac{dt}{1+\alpha^2\sin^2 t}.$$

$u=ctgt$ almashtirish bajaramiz, u holda

$$\int_0^{\frac{\pi}{2}} \frac{dt}{1+\alpha^2 \sin^2 t} = \int_0^{\infty} \frac{du}{1+\alpha^2+u^2},$$

chunki, $du = -\frac{dt}{\sin^2 t}$, $0 \le t \le \frac{\pi}{2}$, $0 \le u < +\infty$. Lekin,

$$\int_0^{+\infty} \frac{du}{1+\alpha^2+u^2} = \int_0^{+\infty} \frac{du}{a^2+u^2} = \frac{1}{a} arctg \frac{u}{a}\Big|_0^{+\infty} = \frac{\pi}{2a},$$

bu yerda $a^2 = \left(\sqrt{1+\alpha^2}\right)^2$. Bu yerdan $I'(\alpha) = \frac{\pi}{2\sqrt{1+\alpha^2}}$ va $I(\alpha) = \int \frac{\pi d\alpha}{2\sqrt{1+\alpha^2}} + C$.

$I(\alpha) = \frac{\pi}{2}\ln(\alpha + \sqrt{1+\alpha^2}) + C$, $I(0) = 0$, $0 = \ln 1 + C$, $C = 0$.

$$I(\alpha) = \int_0^1 \frac{arctg(\alpha x)}{x\sqrt{1-x^2}} dx = \frac{\pi}{2}\ln(\alpha + \sqrt{1+\alpha^2}).$$

5-misol. Ushbu $I = \int_0^{\frac{\pi}{2}} \frac{(x\cos x - 1)(\cos x - x\sin x)}{\ln(x\cos x)} dx$ integralni hisoblang.

Yechish. Yordamchi

$$\Phi(x) = \int_0^1 (x\cos x)^t dt = \frac{(x\cos x)^t}{\ln(x\cos x)}\Big|_0^1 = \frac{x\cos x - 1}{\ln(x\cos x)}$$

interalni kiritaylik.
6-teoremaga ko‘ra,

$$I = \int_0^{\frac{\pi}{2}} F(x)(\cos x - x\sin x)dx = \int_0^{\frac{\pi}{2}} dx \int_0^1 (x\cos x)^t \cdot (\cos x - x\sin x)dt.$$

Agar

$$\int_0^{\frac{\pi}{2}} (x\cos x)^t \cdot (\cos x - x\sin x)dx = \int_0^{\frac{\pi}{2}} (x\cos x)^t d(x\cos x) = \frac{(x\cos x)^{t+1}}{t+1}\Big|_0^{\frac{\pi}{2}} = 0$$

ekanini hisobga olsak, berilgan integral

$$I = \int_0^1 0\, dt = 0$$

tenglikka ega bo‘lamiz.

Mustaqil yechish uchun misollar.

1. $a,b>0$ uchun $\frac{x^b-x^a}{\ln x}=\int_a^b x^y dy$ dan foydalanib,

$$I=\int_0^1 \sin(\ln\frac{1}{x})\frac{x^b-x^a}{\ln x}dx$$

Integralni hisoblang.

2. Integralni paramertr bo‘yicha differensiallashdan foydalanib hisobang.

$$\int_0^{+\infty}\frac{e^{-\alpha x}-e^{-\beta x}}{x}\cos mxdx$$

3. Ushbu

$$I=\int_0^{\frac{\pi}{2}}\frac{\sin^3 x}{\ln\cos x}dt$$

xosmas integralni hisoblang.

4. Parametr bo‘yicha differensiallashdan foydalanib, ushbu integralni hisoblang.

$$J(a,b)=\int_0^{\frac{\pi}{2}}\ln\left(a\sin^2 x+b^2\cos^2 x\right)dx, a,b>0.$$

5. Integralni hisoblang. $J=\int_0^1 \cos(\ln\frac{1}{x})\frac{x^b-x^a}{\ln x}dx$

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