# MINIMUM DENSITY OF TRIANGULATED PACKING'S THREE DIFFERENT SIZE CIRCLES

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## ABSTRACT

The current work demonstrates an ideal packing on a triangular flat torus, particularly those with three tangent circles of varying sizes. We ascertain the minimum and maximum densest packing for three disks of various sizes on a torus. The ratio of a triangle's total area to a circle's sectors on a torus yields the minimal density.

**Keywords:** Tangent Circles**,** Inscribed Circle, Triangulated packing, density, minimum density, disk, flat torus.

## 1. Introduction

Many complex problems in science and engineering involve graphs that are circular and tangent to each other. When a disk packing's contact graph, which is created by connecting the centers of all adjacent disks, only has triangle faces, the packing is referred to be compact or triangulated. A packing's density is defined as the ratio of the torus' area divided by the sum of the disks' areas.

Numerology, granular materials, algebraic number theory, and who knows what else all place a high value on packings of three different size disks. In addition, the Heron's formula is significant and can offer analytical tools, particularly for determining the minimum and maximum density of triangulated packing.

One typical assumption from this kind of mathematical technique is the use of lowest density tangent circles, which is especially useful in situations when maximum densities are difficult to determine.

Now we make a triad of three tangent circles in four distinct methods. First, remove one of the circles, leaving two that tangent to the third. Once all three are tangents, one of the two is rotated around the third (internally or externally), as shown in Fig. 1. In Fig.2 there is one such triad. A triangle is formed by joining the centers $c_1, c_2$, and $c_3$, with side $r_1+r_2$ positioned across from vertex $c_1$, etc.

We can search through these packings to find ones that enhance Heron's formula and go closer to lowest density, which makes the challenge of finding all triangulated packings with three different radii fascinating.

It is critical to understand how close we can get to inscribing the Circle of the Triangle because it aids in answering a more general question: assume that you are sitting in a triangular garden. How would you find its area? Here, the fundamental idea behind the current minimum density of triangulated packing is given in Section 4, followed by an analysis of convergence. The result is presented in Section 5, and the conclusion is in the last Section 6.

## 2. Three Tangent Circles To Each Other

Three circles with radii 1,2 and 3 are tangential to one another, we will find radius of the circle passing through the tangent points, as usual, try to solve the problem before we proceed [1].

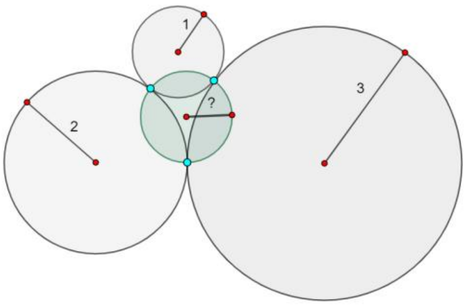


Figure1. Three tangent Circles [1].

Now we can make a proper drawing for the problem so, we have three circles with centers at $c_1$, $c_2$ and $c_3$ they are tangent to one another, the centers form a triangle $\Delta c_1c_2c_3$. As we've seen before, tangent points $T_1$, $T_2$ and $T_3$ lie down on the triangle's sides. Suppose $r_1$, $r_2$ and $r_3$ be the values of the radii, points $T_2$ and $T_3$ lie on the circumference of the firs circle, therefore $c_1T_2 = c_1T_3 = r_1$. Similar equations can be written for other circles [2].

$$c_2T_1 = c_2T_3 = r_2, \quad c_3T_1 = c_1T_2 = r_3.$$

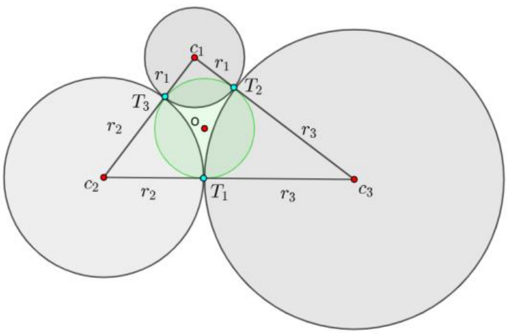


Figure 2. Three tangent circle centers form a triangle [2].

### 3. Finding the Radius of Inscribed Circle

Now we can remove the circles, as they are no longer needed.

Our objective is to find the radius of green circle, which passes through points $T_1$, $T_2$ and $T_3$.

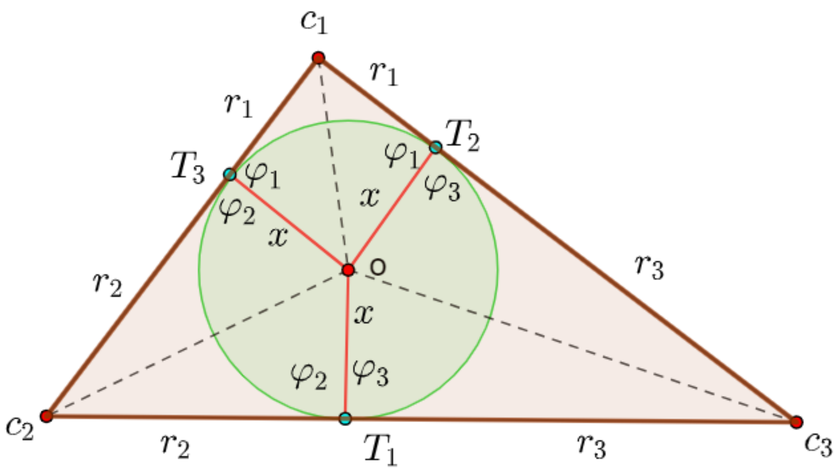


Figure 3. The Inscribed Circle and it reduce [3].

Let $o$ be the center of green circle and $x$ the value of its radius, draw segments $oT_1$, $oT_2$ and $oT_3$, the radii of the green circle are represented by these segments, therefor their lengths are equal to $x$ [4].

$$|oT_1| = |oT_2| = |oT_3| = x.$$

Consider triangles $\Delta c_1oT_2$ and $\Delta c_1oT_3$, they have two pairs of equal sides:

$$|c_1T_2| = |c_1T_3| = r_1, \quad |oT_2| = |oT_3| = x.$$

$\overline{c_1o}$ is the common side of these triangles, therefore the triangles are equal by three sides, from which it follows that angle $\angle oc_1T_2$ equals angle $\angle oc_1T_3$, which means that $\overline{c_1o}$ is the bisector of angle $\angle c_1c_2c_3$. Another pair of equal angles are

$$\angle oT_3c_1 = \angle oT_2c_1.$$

Let $\varphi_1$ be the value of those angles [5].

$$\angle oT_3c_1 = \angle oT_2c_1 = \varphi_1 .$$

In the similar way we prove that $\Delta c_2oT_1 = \Delta c_2oT_3$, from which we conclude that angle $\angle oc_2T_3$, in other words $\overline{c_2o}$ is the bisector of angle $\angle c_1c_2c_3$. We found that $o$ is the point where two bisectors of triangle $\Delta c_1c_2c_3$ meet, therefore $o$ is the incentre of the triangle. This suggests that the green circle may be the inscribed circle of triangle $\Delta c_1c_2c_3$. However, an infinite number of circles are shown there with center in $o$, therefore it is not obvious that the green circle is the inscribed one. Another pair of equal angles are $\angle oT_1c_2 = \angle oT_3c_2$, This will be value $\varphi_2$ [1].

$$\angle oT_1c_2 = \angle oT_3c_2 = \varphi_2$$

Finally, we prove that triangle $\Delta c_3oT_1 = \Delta c_3oT_2$. We get that $\overline{c_3o}$ is the bisector of angle $\angle c_1c_2c_3$. This is actually redundant, as we already know that point $o$ is the incentre of triangle $\Delta c_1c_2c_3$. However, there is another pair of useful equal angles are $\angle oT_1c_3 = \angle oT_2c_3$ and this will be value $\varphi_3$ [6].

$$\angle oT_1c_3 = \angle oT_2c_3 = \varphi_3 .$$

So, it is proven that $o$ is the incentre of triangle $\Delta c_1c_2c_3$.

**4. Inscribed Circle of Triangle**

Now we are going to prove that the green circle is actually the incircle of triangle $\Delta c_1c_2c_3$. For the purpose we use previously introduced angles $\varphi_1, \varphi_2$ and $\varphi_3$. Consider angles near $T_1$. We see that angle $\varphi_2$ and $\varphi_3$ form together a straight angle, so we write $\varphi_2 + \varphi_3 = 180^\circ$. Similarly, for $T_2$ we get $\varphi_1 + \varphi_3 = 180^\circ$, and for $T_3$ we get $\varphi_1 + \varphi_2 = 180^\circ$.

We see that each angle in the equations appears exactly twice, therefore if we add up all the equations, we get $2(\varphi_1 + \varphi_2 + \varphi_3) = 540^\circ$, therefore $\varphi_1 + \varphi_2 + \varphi_3 = 270^\circ$. Now we can get the value of $\varphi_1$ by subtracting the sum of remaining angles from the sum of all angles [2].

$$\varphi_1 = (\varphi_1 + \varphi_2 + \varphi_3) - (\varphi_2 + \varphi_3)$$

And we get $\varphi_1 = 90^\circ$, the same value has $\varphi_2 = 90^\circ$ and $\varphi_3 = 90^\circ$.

This means that the radii $oT_1$, $oT_2$ and $oT_3$ are perpendicular to corresponding sides of triangle $\Delta c_1c_2c_3$.

$$oT_1 \perp c_2c_3, \quad oT_2 \perp c_1c_3 \quad and \quad oT_3 \perp c_1c_2$$

Therefore, all sides of triangle $c_1, c_2$ and $c_3$ are tangent to the circle which means that the green circle is the triangle's incircle $\Delta c_1c_2c_3$ [7].

Assume that $a, b$ and $d$ are the lengths of the triangle's sides $\Delta c_1c_2c_3$.

$$a = |c_2c_3|, \quad b = |c_1c_3| \quad and \quad d = |c_1c_2|.$$

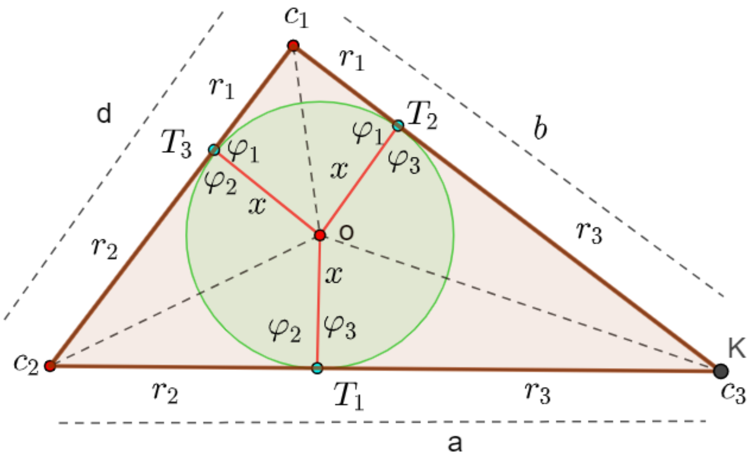


Figure 4. The green circle is actually the incircle of a triangle [1].

There is a simple formula to evaluate the incircle radius for a triangle: $x = \dfrac{A}{s}$, where $A$ denotes the triangle's area $\Delta c_1c_2c_3$, $s$ is semiparametric: $s = \dfrac{a+b+d}{2}$.

To calculate the triangle's area, we will use Heron’s formula

$$A = \sqrt{s(s-a)(s-b)(s-d)}$$

So, we substitute the area from Heron’s formula in to the equation for the radius [1].

$$x = \frac{A}{s} = \sqrt{\frac{s(s-a)(s-b)(s-d)}{s^2}}$$

In the denominator I moved $s$ under the square root, it reduces with $s$ in the numerator and we get the formula for the radius of incircle.

$$x = \sqrt{\frac{(s-a)(s-b)(s-d)}{s}}$$

Now we need to express triangle's sides in terms of radii $r_1, r_2$ and $r_3$. It can be easily seen that $a = r_2 + r_3, \quad b = r_1 + r_3, \quad d = r_1 + r_2.$

Then we evaluate the semiparametric, where each radius appears exactly twice, then factors 2 are cancelled out and we get just the sum of radii.

$$s = \frac{(r_2 + r_3) + (r_1 + r_3) + (r_1 + r_2)}{2} = \frac{2(r_1 + r_2 + r_3)}{2} = r_1 + r_2 + r$$

Next evaluate the other factors in Heron's formula [4].

$$s - a = (r_1 + r_2 + r_3) - (r_2 + r_3)$$

For $s - a$ the radii are nicely cancelled out and we get just $r_1$.

$$s - a = (r_1 + r_2 + r_3 - r_2 - r_3) = r_1$$

In the similar way, $s - b = r_2$ *and* $s - d = r_3$ and there is the final formula for the radius of the green circle.

$$x = \sqrt{\frac{r_1 r_2 r_3}{r_1 + r_2 + r_3}}.$$

Substituting $r_1 = 1, \ r_2 = 2, \ r_3 = 3$ in to the formula for the radius, gives:

$$x = \sqrt{\frac{r_1 r_2 r_3}{r_1 + r_2 + r_3}} = \sqrt{\frac{1 \times 2 \times 3}{1 + 2 + 3}} = 1.$$

However, in this particular case we can get away without Heron's formula. Calculating the triangle's sides gives: $a = r_2 + r_3 = 2 + 3 = 5, \quad b = r_1 + r_3 = 1 + 3 = 4$ and $d = r_1 + r_2 = 1 + 2 = 3$.

This is the famous Egyptian triangle, which is a right-angled triangle. Therefore, the area is just half product of the sides, which is 6,

$$A = \frac{1}{2} bd = \frac{3 \times 4}{2} = 6$$

And we get 6 for the semiparametric as well.

$$s = \frac{a + b + d}{2} = \frac{3 + 4 + 5}{2} = 6.$$

Therefore, the radius is again 1.

$$x = \frac{A}{s} = \frac{6}{6} = 1.$$

## 5. Results and Discussion

### 5.1. Minimum and Maximum Density

We can show that the all-triangulated packings' density is minimum if all the radii of all the disks are equal.

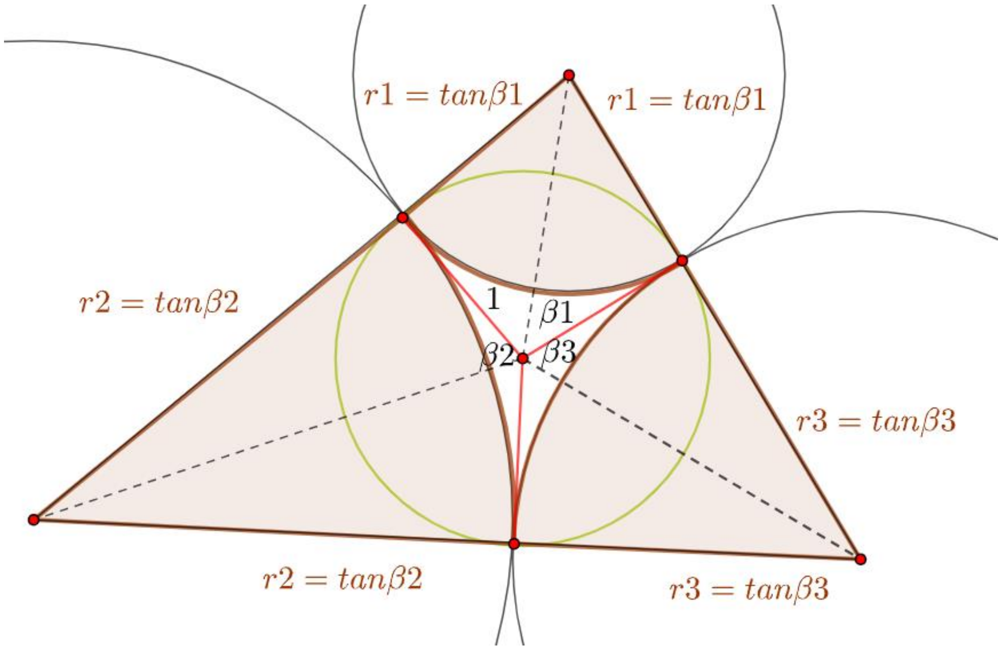


Figure 5. shows a triangle formed by three mutually tangent circles that formed of connecting the centers of three circles. There are additionally angle bisectors for the triangle and the circle it has inscribed [8].

The incircle's radius is one, because the triangle is normalized. The incircle's radii are orthogonal to the triangle's sides. The angles between the bisectors are shown as $\beta_1$, $\beta_2$ and $\beta_3$ a red section adjacent to it and each of $\tan\beta_i = r_i$ ,the $i$-is the radius of circle. The packing density in the triangle, the area of the grey region divided by the total area of the triangle is equal to $\delta(\beta_1, \beta_2, \beta_3)$. When the three circles' radii are the same size then the $\delta$ is minimized. We will show it as follow:

As shown in Figure 5, let $A(\beta_1, \beta_2, \beta_3)$ be the grey sector's union area [9]. Let

$$A_1(\beta_1) = \frac{1}{2}\left(\frac{\pi}{2} - \beta_1\right) r_1^2 + \frac{1}{2}\left(\frac{\pi}{2} - \beta_1\right) r_1^2 = \left(\frac{\pi}{2} - \beta_1\right) r_1^2 = \left(\frac{\pi}{2} - \beta_1\right) \tan^2(\beta_1),$$

$$A_2(\beta_2) = \frac{1}{2}\left(\frac{\pi}{2} - \beta_2\right) r_2^2 + \frac{1}{2}\left(\frac{\pi}{2} - \beta_2\right) r_2^2 = \left(\frac{\pi}{2} - \beta_2\right) r_2^2 = \left(\frac{\pi}{2} - \beta_2\right) \tan^2(\beta_2),$$

$$A_3(\beta_3) = \frac{1}{2}\left(\frac{\pi}{2} - \beta_3\right) r_3^2 + \frac{1}{2}\left(\frac{\pi}{2} - \beta_3\right) r_3^2 = \left(\frac{\pi}{2} - \beta_3\right) r_3^2 = \left(\frac{\pi}{2} - \beta_3\right) \tan^2(\beta_3).$$

$$A(\beta_1,\beta_2,\beta_3)=A_1(\beta_1)+A_2(\beta_2)+A_3(\beta_3)$$
$$=\left(\frac{\pi}{2}-\beta_1\right)\tan^2(\beta_1)+\left(\frac{\pi}{2}-\beta_2\right)\tan^2(\beta_2)+\left(\frac{\pi}{2}-\beta_3\right)\tan^2(\beta_3) \qquad (1)$$

The length of a side equal to a right triangle with twice the area is 1. and angle $\beta$ adjacent to that unit length is $\tan(\beta)$.

$\beta$ is the angle adjacent to that unit length, and $\tan(\beta)$ is the angle adjacent to that unit length? Assume $S(\beta_1,\beta_2,\beta_3)$ is the triangle's area as shown in Figure [10]. The area of the triangle is equal to the sum of the six smaller right triangles' areas, therefore

$$S_1=\frac{1}{2}r_1\cdot 1+\frac{1}{2}r_1\cdot 1=r_1\cdot 1=\tan\beta_1\cdot 1=\tan\beta_1,$$
$$S_2=\frac{1}{2}r_2\cdot 1+\frac{1}{2}r_2\cdot 1=r_2\cdot 1=\tan\beta_2\cdot 1=\tan\beta_2,$$
$$S_3=\frac{1}{2}r_3\cdot 1+\frac{1}{2}r_3\cdot 1=r_3\cdot 1=\tan\beta_3\cdot 1=\tan\beta_3.$$

We obtain the total amount of $S_1,\ S_2\ and\ \ S_3$,

$$S(\beta_1,\beta_2,\beta_3)=S_1+S_2+S_3=\tan(\beta_1)+\tan(\beta_2)+\tan(\beta_3) \qquad (2)$$

As a result, the density of the triangle's covered portion, as indicated in the figure 5,
is

$$\delta(\beta_1,\beta_2,\beta_3)=\frac{A(\beta_1,\beta_2,\beta_3)}{S(\beta_1,\beta_2,\beta_3)}=\frac{\left(\frac{\pi}{2}-\beta_1\right)\tan^2(\beta_1)+\left(\frac{\pi}{2}-\beta_2\right)\tan^2(\beta_2)+\left(\frac{\pi}{2}-\beta_3\right)\tan^2(\beta_3)}{\tan(\beta_1)+\tan(\beta_2)+\tan(\beta_3)} \qquad (3)$$

We'll suppose that $\beta_1+\beta_2+\beta_3=\pi$ and each $0<\beta_i<\frac{\pi}{2}$ are in this case, and that the angles derive from the condition depicted in Figure5 [3].

**Theorem1.** Density of packing $\delta$ has a minimum value of $\frac{\pi}{\sqrt{12}}$, and is obtained only when

$$\beta_1=\beta_2=\beta_3=\frac{\pi}{3}.$$

Proof. We use formula of (3) and we get

$$\delta(\beta_1,\beta_2,\beta_3)=\frac{\left(\frac{\pi}{2}-\beta_1\right)\tan^2(\beta_1)+\left(\frac{\pi}{2}-\beta_2\right)\tan^2(\beta_2)+\left(\frac{\pi}{2}-\beta_3\right)\tan^2(\beta_3)}{\tan(\beta_1)+\tan(\beta_2)+\tan(\beta_3)}$$

$$\delta=\frac{\left(\frac{\pi}{2}-\frac{\pi}{3}\right)\tan^2(\frac{\pi}{3})+\left(\frac{\pi}{2}-\frac{\pi}{3}\right)\tan^2(\frac{\pi}{3})+\left(\frac{\pi}{2}-\frac{\pi}{3}\right)\tan^2(\frac{\pi}{3})}{\tan(\frac{\pi}{3})+\tan(\frac{\pi}{3})+\tan(\frac{\pi}{3})}$$
$$=\frac{\frac{\pi}{6}\left(\sqrt{3}\right)^2+\frac{\pi}{6}\left(\sqrt{3}\right)^2+\frac{\pi}{6}\left(\sqrt{3}\right)^2}{\sqrt{3}+\sqrt{3}+\sqrt{3}}=\frac{\frac{3\pi}{2}}{3\sqrt{3}}=\frac{\pi}{\sqrt{12}}.$$

Let the contacting circles to be equal, and the $\tan\beta_i$ for our normalization, then this minimum density is obtained. To simplify the calculations, instead of simply calculating the critical minimum density, we can compute the complementary maximum density [6].

$$\bar{\delta}=1-\delta=\frac{S(\beta_1,\beta_2,\beta_3)-A(\beta_1,\beta_2,\beta_3)}{S(\beta_1,\beta_2,\beta_3)}.$$

Finally, we abstained the minimum and maximum density in a triangulated packing.

5.2. Density in Terms of Radii

Considering the three radii that make up the triangle, it is possible to write with the same density. The area of a triangle is

$$T_r=T_r(r_1,r_2,r_3)=\sqrt{r_1r_2r_3\left(r_1+r_2+r_3\right)},$$

according to Heron's formula [11].

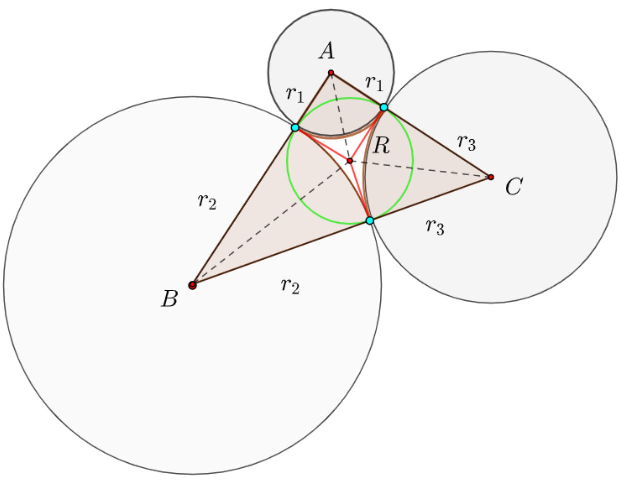


Figure 6: Density in Terms of Radii [11].

To compute the triangle's area, we will utilize Heron's formula

$$T = \sqrt{s(s-a)(s-b)(s-c)}$$

Where $s$ is semiparametric of triangle $\Delta ABC$ and we obtain as follow:

$$s = \frac{a+b+c}{2}$$

And also $\overline{AB} = r_1 + r_2 = c$, $\overline{BC} = r_2 + r_3 = a$ *and* $\overline{AC} = r_1 + r_3 = b$

Then we evaluate the semiparametric, where each radius appears exactly twice, then we get just the sum of radii [12].

$$s = \frac{(r_2+r_3)+(r_1+r_3)+(r_1+r_2)}{2} = \frac{2(r_1+r_2+r_3)}{2} = r_1+r_2+r_3$$

Next evaluate the other factors in Heron's formula.

$$s-a = (r_1+r_2+r_3)-(r_2+r_3)$$

For $s-a$ the radii are nicely cancelled out and we get just $r_1$ [9].

$$s-a = (r_1+r_2+r_3-r_2-r_3) = r_1$$

In the similar way,

$$s-b = (r_1+r_2+r_3)-(r_1+r_3) = r_2 \quad and \quad s-c = (r_1+r_2+r_3)-(r_1+r_2) = r_3.$$

Then substitute in Heron's formula and we get area of the triangle as follow

$$T_r = T_r(r_1, r_2, r_3) = \sqrt{r_1 r_2 r_3 (r_1+r_2+r_3)},$$

where $R$ is the inradius (The incircle's radius has been standardized to 1 in the diagram). There is a simple formula to evaluate the incircle radius for a triangle:

$$R = \frac{T}{s},$$

Thus, the inradius is

$$R = R(r_1, r_2, r_3) = \frac{T}{s} = \sqrt{\frac{(s-a)(s-b)(s-c)}{s^2}} = \sqrt{\frac{r_1 r_2 r_3 \left(r_1 + r_2 + r_3\right)}{\left(r_1 + r_2 + r_3\right)^2}} = \sqrt{\frac{r_1 r_2 r_3}{r_1 + r_2 + r_3}}$$

If $r_1 = 1,\ r_2 = 2 \quad and \quad r_3 = 3$ then

$$R = \sqrt{\frac{r_1 r_2 r_3}{r_1 + r_2 + r_3}} = \sqrt{\frac{1 \times 2 \times 3}{1 + 2 + 3}} = 1$$

As we have area of sector $= \frac{1}{2}\theta r^2$

Let the area of the union of the grey sectors be $Q(r_1, r_2, r_3)$ as in Figure 6 [10]. As shown in Figure, the combined area of the grey sectors is $Q(r_1, r_2, r_3)$. Let

$$Q_1(r_1) = \frac{1}{2}\left(\frac{\pi}{2} - \tan^{-1}(r_1 / R)\right) r_1^2 + \frac{1}{2}\left(\frac{\pi}{2} - \tan^{-1}(r_1 / R)\right) r_1^2 = \left(\frac{\pi}{2} - \tan^{-1}(r_1 / R)\right) r_1^2,$$

$$Q_2(r_2) = \frac{1}{2}\left(\frac{\pi}{2} - \tan^{-1}(r_2 / R)\right) r_2^2 + \frac{1}{2}\left(\frac{\pi}{2} - \tan^{-1}(r_2 / R)\right) r_2^2 = \left(\frac{\pi}{2} - \tan^{-1}(r_2 / R)\right) r_2^2,$$

$$Q_3(r_3) = \frac{1}{2}\left(\frac{\pi}{2} - \tan^{-1}(r_3 / R)\right) r_3^2 + \frac{1}{2}\left(\frac{\pi}{2} - \tan^{-1}(r_3 / R)\right) r_3^2 = \left(\frac{\pi}{2} - \tan^{-1}(r_3 / R)\right) r_3^2.$$

Then we have

$$\begin{aligned} Q(r_1, r_2, r_3) &= Q_1(r_1) + Q_2(r_2) + Q_3(r_3) \\ &= \left(\frac{\pi}{2} - \tan^{-1}(r_1 / R)\right) r_1^2 + \left(\frac{\pi}{2} - \tan^{-1}(r_2 / R)\right) r_2^2 + \left(\frac{\pi}{2} - \tan^{-1}(r_3 / R)\right) r_3^2 \end{aligned}$$

As a result, the three disks' density arranged in a triangle found by Connelly [3]as in Figure is

$$\delta_r(r_1, r_2, r_3) = \frac{\left(\frac{\pi}{2} - \tan^{-1}(r_1 / R)\right) r_1^2 + \left(\frac{\pi}{2} - \tan^{-1}(r_2 / R)\right) r_2^2 + \left(\frac{\pi}{2} - \tan^{-1}(r_3 / R)\right) r_3^2}{T_r(r_1, r_2, r_3)}$$

Or

$$\delta_r(r_1, r_2, r_3) = \frac{\left(\frac{\pi}{2} - \tan^{-1}(r_1 / R)\right) r_1^2 + \left(\frac{\pi}{2} - \tan^{-1}(r_2 / R)\right) r_2^2 + \left(\frac{\pi}{2} - \tan^{-1}(r_3 / R)\right) r_3^2}{\sqrt{r_1 r_2 r_3 (r_1 + r_2 + r_3)}}$$

.

## 6. Conclusions

In order to calculate the radius of the inscribed circle and the triangle's inscribed circle, three tangent circles were investigated. We drew an appropriate diagram for the issue after plotting the diagram of the three circles with radii of 1, 2, and 3, and then we eliminated the circles because they were no longer necessary. We demonstrated how the triangle's inner circle is represented by the green circle. In addition, we demonstrated that the inscribed circle's radius equals 1.

We plotted a triangle produced by joining the centers of the three circles of the triangulated packing minimum density as well as a diagram of three mutually tangent circles to summarize the findings and give an overview of them.

We calculated the area of the union of the grey sectors $A(\beta_1, \beta_2, \beta_3)$ and the area of the normalized triangle $S(\beta_1, \beta_2, \beta_3)$ in order to determine the density of the packing in the triangle. As a result, in (3), we arrived at the density of the covered portion of the triangle as shown in Figure 5.

In some instances, where all of the disks' radii are equal, we can achieve the least density of all triangulated packings. The final value of $\delta$ is $\frac{\pi}{\sqrt{12}}$, which can only be reached when $\beta_1 = \beta_2 = \beta_3 = \frac{\pi}{3}$.

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