UCHINCHI TARTIBLI TUZILMALI TURDAGI TENGLAMA UCHUN CHEGARAVIY MASALANING SHARTLI KORREKTLIGI

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ANNOTATSIYA

Ushbu maqolada uchinchi tartibli tuzilmali turdagi xususiy hosilali differensial tenglama uchun boshlang‘ich-chegaraviy nokorrekt masala qaralgan. Bunda masalaning nokorrektligi ko‘rsatilgan. Integral energiya usuli bilan aprior baho olingan. Korrektlik to‘plamida yechimning yagonalik va shartli turg‘unlik teoremalari isbotlangan.

Kalit so‘zlar: nokorrekt masala, tuzilmali tenglama, aprior baho, yagonalik teoremasi, shartli turg’unlik teoremasi.

Maqola uchinchi tartibli tuzilmali turdagi xususiy hosilali differensial tenglama uchun nokorrekt qo‘yilgan boshlang‘ich-chegaraviy masalani o‘rganishga bag‘ishlangan.

Berilgan $\Omega=\{0<x<\pi,\, 0<t<T, T>0\}$ sohada

$$\left(\frac{\partial}{\partial t}+I\right)\left(\frac{\partial^2}{\partial t^2}+a^2\frac{\partial^2}{\partial x^2}+b\right)u(x,t)=0 \qquad (1)$$

tenglamani qaraymiz, bunda I - birlik operator, a,b - berilgan haqiqiy sonlar, $a\neq 0$.

Masala. (1) tenglamani hamda boshlang'ich

$$u|_{t=0}=\varphi(x),\, u_t|_{t=0}=\psi(x),\, u_{tt}|_{t=0}=\phi(x),\ 0\le x\le\pi, \qquad (2)$$

chegaraviy

$$u|_{x=0}=u|_{x=\pi}=0\ ,\ 0\le t\le T\ , \qquad (3)$$

shartlarni qanoatlantiruvchi $u(x,t)$ funksiyani toping.

Elliptik turdagi tenglama uchun nokorrekt qo‘yilgan masalalar F.John [1], M.M.Lavrent’ev [4], S.G. Kreyn [3], M.Landis [6], V.K.Ivanov [2] va boshqalarning tadqiqot ob’ekti bo‘lgan.

Tuzilmali va aralash-tuzilmali turdagi tenglamalar uchun nokorrekt masalalar [7, 8] ilmiy maqolalarda o‘rganilgan.

Amaliyotning dolzarb vazifalaridan biri yuqori tartibli xususiy hosilali differensial tenglamalar uchun nokorrekt qo‘yilgan chegaraviy masalalarning regulyarlashgan yechimini topish hisoblanadi. Nokorrekt masalalarning taqribiy regulyarlashgan yechimi qurish uchun, ushbu masalaning yechimi yagonaligi va shartli turg’unligi haqidagi teoremalar isbotlanishi kerak.

Ushbu ishda (1)-(3) nokorrekt masala yechimining mos korrektlik to’plamda yagonalik, shartli turg’unlik teoremalari isbotlangan.

Ta'rif. (1)-(4) masalaning yechimi deganda (1) tenglamada qatnashuvchi uzluksiz hosilalarga ega, (2)-(3) shartlarni va Ω sohada (1) tenglamani qanoatlantiradigan $u(x,t)$ funksiya tushuniladi.

Masalaning nokorrektligi. Aytaylik $\varphi(x)=0, \phi(x)=0$ va $\psi(x)$ funksiya esa $\psi_n(x)=\frac{1}{n}\sin nx$ bo‘lsin, $n\in N$. U holda (1)-(3) masalaning yechimi

$$u_n(x,t)=\frac{1}{n}\int_0^t e^{\tau-t}ch\left(\sqrt{a^2n^2-b}\,\tau\right)d\tau\cdot\sin nx$$

ko‘rinishga ega bo‘ladi. Bu yerdan, ma’lumki $n\to\infty$ bo‘lganda $\max_x|\psi_n(x)|\to 0$ bo‘ladi, bu vaqtda $\max_x|u_n(x,y)|\to\infty$ ekanligi oson kelib chiqadi. Bu esa korrektlikning turg'unlik sharti buzilganligini ko'rsatadi, ya'ni yechimning Koshi berilganlariga doimiy bog‘liqligi yo'q.

1-lemma. Faraz qilamiz $u(x,t)$ funksiya Ω sohada (1) tenglamani va (2), (3) shartlarni qanoatlantirsin. U holda $u(x,t)$ funksiya uchun

$$\|u\|^2\le 2\left(\|\varphi(x)\|^2+t\int_0^t\left(\left(\|\psi(x)+\varphi(x)\|^2+|\alpha|\right)^{1-\frac{\tau}{T}}\left(\|u_t(x,T)+u(x,T)\|^2+|\alpha|\right)^{\frac{\tau}{T}}\cdot e^{2\tau(t-T)}\right)d\tau\right)$$

tengsizlik o‘rinli, bu yerda

$$\alpha=\frac{1}{2}\int_0^\pi\left(a^2(\psi'(x)+\varphi'(x))^2-b(\psi(x)+\varphi(x))^2-(\phi(x)+\psi(x))^2\right)dx.$$

Isbot. (1)-(3) masalada $\frac{\partial u}{\partial t}+u=v$ belgilash kiritilsa $u(x,t)$ va $v(x,t)$ funksiyalar uchun mos ravishda quyidagi masalalarga kelamiz:

$$\begin{cases} u_t+u=v,\ 0<x<\pi,\ 0<t<T, \\ u|_{t=0}=\varphi(x), 0\le x\le\pi \end{cases} \quad (4)$$

va

$$\begin{cases} v_{tt} + a^2 v_{xx} + bv = 0, 0 < x < \pi,\ 0 < t < T, \\ v\big|_{t=0} = \alpha(x),\ 0 \le x \le \pi, \\ v_t\big|_{t=0} = \beta(x),\ 0 \le x \le \pi, \\ v\big|_{x=0} = v\big|_{x=\pi} = 0, 0 \le t \le T \end{cases} \quad (5)$$

bu yerda $\alpha(x) = \psi(x) + \varphi(x)$, $\beta(x) = \phi(x) + \psi(x)$.

Endi (5) masalaning yechimi uchun $f(t) = \int_0^\pi v^2 dx$ funksiyani qaraymiz. Bundan

$$f'(t) = 2\int_0^\pi v \cdot v_t dx,\ \ f''(t) = 2\int_0^\pi v_t^2 dx + 2\int_0^\pi v \cdot v_{tt} dx$$

bo‘lishini topamiz. $f''(t)$ ifodaning ikkinchi hadi

$$I_2 = \int_0^\pi v \cdot v_{tt} dx = -\int_0^\pi v(a^2 v_{xx} + bv) dx = -a^2 \int_0^\pi v \cdot v_{xx} dx - b\int_0^\pi v^2 dx .$$

Differensial olamiz

$$\frac{dI_2}{dt} = -a^2 \int_0^\pi v_t \cdot v_{xx} dx - a^2 \int_0^\pi v \cdot v_{xxt} dx - 2b\int_0^\pi v \cdot v_t dx = 2a^2 \int_0^\pi v_t \cdot v_{xx} dx - 2b\int_0^\pi v \cdot v_t dx .$$

Bu yerda (5) masalaning chegaraviy shartlari hisobiga $\int_0^\pi v \cdot v_{xxt} dx = \int_0^\pi v_{xx} \cdot v_t dx$ tenglikdan foydalanildi. Demak,

$$-2\left(\int_0^\pi a^2 v_t v_{xx} dx + \int_0^\pi bv \cdot v_t dx \right) = -2\int_0^\pi (a^2 v_{xx} dx + bv) v_t dx = 2\int_0^\pi v_{tt} \cdot v_t dx = \frac{d}{dt}\left(\int_0^\pi v_t^2 dx \right).$$

Yuqoridagilardan

$$\frac{d}{dt}\left(a^2 \int_0^\pi v_x^2 dx - b\int_0^\pi v^2 dx \right) = \frac{d}{dt}\left(\int_0^\pi v_t^2 dx \right).$$

Bu tenglikni integrallab

$$a^2 \int_0^\pi v_x^2 dx - b\int_0^\pi v^2 dx = \int_0^\pi v_t^2 dx + 2\alpha$$

tenglikka ega bo‘lamiz, bu yerda $\alpha = \frac{1}{2}\int_0^\pi \left(a^2 v_x^2 - bv^2 - v_t^2 \right)_{t=0} dx$. Natijada

$$f''(t) = 4\int_0^\pi v_t^2 dx + 4\alpha .$$

Endi $g(t)=\ln\left(f(t)+|\alpha|\right)$ belgilash kiritamiz. Ma'lumki

$$g''(t)=\frac{f''(t)(f(t)+|\alpha|)-f'^2(t)}{(f(t)+|\alpha|)^2}=\frac{\left(4\int_0^\pi v_t^2dx+4\alpha\right)\left(\int_0^\pi v^2dx+|\alpha|\right)-\left(2\int_0^\pi v\cdot v_tdx\right)^2}{\left(\int_0^\pi v^2dx+|\alpha|\right)^2}$$

$$=\frac{4\int_0^\pi v_t^2dx\int_0^\pi v^2dx+4|\alpha|\int_0^\pi v_t^2dx+4\alpha\left(\int_0^\pi v^2dx+|\alpha|\right)-4\left(\int_0^\pi v\cdot v_tdx\right)^2}{\left(\int_0^\pi v^2dx+|\alpha|\right)^2}\geq$$

$$\geq\frac{4\alpha\left(\int_0^\pi v_t^2dx+|\alpha|\right)}{\left(\int_0^\pi v^2dx+|\alpha|\right)^2}=\frac{4\alpha}{\int_0^\pi v^2dx+|\alpha|}\geq -4.$$

Demak $g''(t)+4\geq 0$ tengsizlikdan, ya'ni logarifmik qavariq funksiya xususiyatlaridan foydalanib

$$\int_0^\pi v^2dx\leq\left(\int_0^\pi v^2\Big|_{t=0}dx+|\alpha|\right)^{1-\frac{t}{T}}\left(\int_0^\pi v^2\Big|_{t=T}dx+|\alpha|\right)^{\frac{t}{T}}e^{2t(t-T)}-|\alpha| \qquad (6)$$

tengsizlikka ega bo'lamiz. (4) masala yechimi

$$u(x,t)=\varphi(x)+\int_0^t v(x,\tau)\cdot e^{\tau-t}d\tau .$$

Bundan esa

$$\|u\|^2\leq 2\left(\|\varphi(x)\|^2+t\int_0^t\|v\|^2d\tau\right)$$

baho kelib chiqadi. (6) tengsizlik va (4), (5) masala shartlaridan talab qilingan tengsizlik kelib chiqadi.

(1)-(3) masalaning korrektlik to'plamini quyidagicha kiritamiz:

$$M=\left\{u(x,t):\|u_t(x,T)+u(x,T)\|\leq m\right\}.$$

Teorema 1. Faraz qilamiz (1) – (3) masalaning yechimi mavjud va $u\in M$ bo'lsin. U holda (1) – (3) masala yechimi yagonadir.

Isbot. Faraz qilamiz (1) – (3) masalaning yechimi ikkita bo‘lsin, ya’ni $u_1(x,t)$ va $u_2(x,t)$. $U(x,t)=u_1(x,t)-u_2(x,t)$ belgilash kiritamiz. U holda $U(x,t)$funksiya

$$\left(\frac{\partial}{\partial t}+I\right)\left(\frac{\partial^2}{\partial t^2}+a^2\frac{\partial^2}{\partial x^2}+b\right)U=0$$

tenglamani,

$$U\big|_{t=0}=0,\ \ U_t\big|_{t=0}=0,\ \ U_{tt}\big|_{t=0}=0$$

Boshlang‘ich shartlarni va

$$U\big|_{x=0}=U\big|_{x=\pi}=0$$

chegaraviy shartlarni qanoatlantiradi.

Demak, $U(x,t)$ funksiya uchun 1-lemma natijasiga ko’ra $\|U\|^2\le 0$. Bundan $\|U\|=0$ hosil bo‘ladi. Bundan esa $u_1\equiv u_2$ bo‘lishi, yoki (1)-(3) masala yechimi yagonaligi kelib chiqadi.

Faraz qilamiz $u(x,t)$ funksiya (1)-(3) masalada $\varphi(x)$, $\psi(x)$ va $\phi(x)$ aniq berilganlarga mos yechim, $u_\varepsilon(x,t)$ funksiya esa $\varphi_\varepsilon(x)$, $\psi_\varepsilon(x)$ va $\phi_\varepsilon(x)$ taqribiy berilganlarga mos yechim bo‘lsin.

Teorema 2. Faraz qilamiz (1) - (3) masalaning yechimi mavjud, $u,u_\varepsilon\in M$ $\|\varphi(x)-\varphi_\varepsilon(x)\|\le\varepsilon,\ \|\psi(x)-\psi_\varepsilon(x)\|\le\varepsilon,\ \|\phi(x)-\phi_\varepsilon(x)\|\le\varepsilon$ bo‘lsin, $\varepsilon>0$. U holda

$$\|u-u_\varepsilon\|^2\le 2\left(\varepsilon^2+t\int_0^t\left(2(a^2+b+3)\varepsilon^2\right)^{1-\frac{\tau}{T}}\cdot\left(4m^2+2(a^2+b+1)\varepsilon^2\right)^{\frac{\tau}{T}}e^{2\tau(\tau-T)}d\tau\right)$$

tengsizlik o’rinli.

Isbot. $U(x,t)=u(x,t)-u_\varepsilon(x,t)$ belgilash kiritamiz. U holda $U(x,t)$ funksiya

$$\left(\frac{\partial}{\partial t}+I\right)\left(\frac{\partial^2}{\partial t^2}+a^2\frac{\partial^2}{\partial x^2}+b\right)U=0,\ (x,t)\in\Omega,$$

$$U(x,t)\big|_{t=0}=\varphi(x)-\varphi_\varepsilon(x),U_t(x,t)\big|_{t=0}=\psi(x)-\psi_\varepsilon(x),U_{tt}(x,t)\big|_{t=0}=\phi(x)-\phi_\varepsilon(x),\ 0\le x\le\pi,$$

$$U(x,t)\big|_{x=0}=U(x,t)\big|_{x=\pi}=0,\ 0\le t\le T$$

masalani qanoatlantiradi. 1-lemmaga asosan $U(x,t)$ funksiya uchun quyidagi tengsizlik o‘rinli

$$\|U\|^2 \le 2\|\varphi(x)-\varphi_\varepsilon(x)\|^2 +$$

$$2t\int_0^t\left(\left(\|\psi(x)-\psi_\varepsilon(x)+\varphi(x)-\varphi_\varepsilon(x)\|^2+|\alpha|\right)^{1-\frac{\tau}{T}}\left(\|U_t(x,T)+U(x,T)\|^2+|\alpha|\right)^{\frac{\tau}{T}}\cdot e^{2\tau(\tau-T)}\right)d\tau$$

bu yerda

$$\alpha=\frac{1}{2}\int_0^\pi\left(a^2\left(U_{xt}+U_x\right)^2-b(U_t+U)^2-(U_{tt}+U_t)^2\right)\Big|_{t=0}dx.$$

Quyidagilarni baholaymiz

$$|\alpha|\le\int_0^\pi\left(a^2U_{xt}^2+a^2U_x^2+bU_t^2+bU^2+U_{tt}^2+U_t^2\right)\Big|_{t=0}dx=$$

$$=a^2\int_0^\pi(\psi'-\psi_\varepsilon')^2dx+a^2\int_0^\pi(\varphi'-\varphi_\varepsilon')^2dx+b\int_0^\pi(\psi-\psi_\varepsilon)^2dx+$$

$$b\int_0^\pi(\varphi-\varphi_\varepsilon)^2dx+\int_0^\pi(\phi-\phi_\varepsilon)^2dx+\int_0^\pi(\psi-\psi_\varepsilon)^2dx\le 2(a^2+b+1)\varepsilon^2,$$

$$\|\psi(x)-\psi_\varepsilon(x)+\varphi(x)-\varphi_\varepsilon(x)\|^2\le 2\|\psi(x)-\psi_\varepsilon(x)\|^2+2\|\varphi(x)-\varphi_\varepsilon(x)\|^2\le 4\varepsilon^2,$$

$$\|U_t(x,T)+U(x,T)\|^2\le\|u_t(x,T)-u_{\varepsilon t}(x,T)+u(x,T)-u_\varepsilon(x,T)\|^2$$

$$\le 2\|u_t(x,T)+u(x,T)\|^2+2\|u_{\varepsilon t}(x,T)+u_\varepsilon(x,T)\|^2=4m^2.$$

Bularni hisobga olib

$$\|U\|^2\le 2\left(\varepsilon^2+t\int_0^t\left(2(a^2+b+3)\varepsilon^2\right)^{1-\frac{\tau}{T}}\cdot\left(4m^2+2(a^2+b+1)\varepsilon^2\right)^{\frac{\tau}{T}}e^{2\tau(\tau-T)}d\tau\right)$$

tengsizlikka ega bo‘lamiz. Bundan esa talab qilingan tengsizlik kelib chiqadi.

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