RESEARCH AROUND ONE WEAK SOLUTION FOR SPECIAL OPERATOR IN VARIABLE EXPONENT SPACES

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ABSTRACT

We prove the existence of one weak solution for the fourth-order problem involving the special operator in variable exponent spaces. The proof of our main result uses variational methods.

This-type problems are used to describe a large class of physical phenomena Such as micro-electro-mechanical systems, phase field models of multiphase Systems, thin film theory, thin plate theory, surface diffusion on Solids, interface dynamics, and also flow in Hele–Shaw cells. That is why many authors have looked for solutions of elliptic equations involving such operators.

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Introduction

In this article, we show the existence of one weak solution for the following Fourth-order problem involving the special type operator

$$\begin{cases} \Delta(a(x, \Delta u) = \lambda |u|^{p-2} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \qquad (1,1)$$

Where Ω is a bounded domain in $\mathbb{R}^N (N \geq 2)$ with a smooth boundary $\partial\Omega$, $\lambda > 0$ is a parameter, u is a Carathéodory function, $p \in C(\overline{\Omega})$ satisfies the inequality

$$\inf_{x \in \Omega} > \frac{N}{2} \qquad \text{for all} \quad x \in \Omega,$$

And $\Delta(a(x, \Delta u)$ is the special operator of the fourth-order, where a satisfies a Set of conditions.

We assume that the $a: \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ satisfy the following conditions:

(F1) a is a Carathéodory function such that $a(x; 0) = 0$, for a. e. $x \in \overline{\Omega}$.

(F2) There exist $c_1 > 0$, such that

$$|a(x, t)| \leq c_1 \left(1 + |t|^{p(x)-1}\right), \quad \text{for a. e } x \in \Omega \text{ and } t \in \mathbb{R}$$

(F3) For all $s; t \in \mathbb{R}$, the inequality $|a(x; t) - a(x; s)|(t - s) \geq 0$ holds, for a. e. $x \in \Omega$.

(F4) There exists $c_2 \geq 1$ such that

$$c_2|t|^{p(x)} \leq \min\{a(x,t)t, p(x)A(x,t)\}, \qquad \text{for a.e.}\, x \in \Omega \text{ and all } t \in \mathbb{R},$$

Where $A: \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ represents the anti-derivative of a, that is,

$$A(x,t) = \int_0^t a(x,s)\,ds$$

Assume conditions (F1) − (F4) hold, then we have

(I) $A(x,t)$ is a C^1-Carathéodory function, i.e. for every $t \in \mathbb{R}$, $A(\cdot,\ t): \Omega \to \mathbb{R}$ is measurable and for a.e. $x \in \Omega$, $A(x,\ \cdot)$ is $C^1(\mathbb{R})$.

(II) There exists a constant c_3 such that

$$|A(x,t)| \leq c_3\left(|t| + |t|^{p(x)}\right), \quad \text{for a.e } x \in \Omega \text{ and all } t \in \mathbb{R}.$$

Details about this kind of operators the reader is referred to [4, 9, 10].
Investigations of this type of operators has been going on in various fields, e.g. in electro rheological fluids (see [13]), elasticity theory (see [15]), stationary thermo rheological viscous flows of non-Newtonian fluids (see [1]), image processing (see [5]), and mathematical description of the processes

filtration of bar tropic gas through a porous medium (see [2]), For more details about this kind of operators the reader is referred to Leray and Lions [10].
For the reader's convenience, we recall some background facts concerning the variable exponent Lebesgue and Sobolev spaces with variable exponent and introduce some notation. For more details, we refer the reader to [6, 7, 8, **9**, 11, **12**] and the references therein. Set

$$C_+(\overline{\Omega}) := \{h \in C(\overline{\Omega}): h(x) > 1, \qquad \text{for all } x \in \overline{\Omega}\}.$$

For every $p \in C_+(\overline{\Omega})$,

$$1 < p^- := \min_{x \in \overline{\Omega}} p(x) \leq \max_{x \in \overline{\Omega}} p(x) < +\infty$$

We define the variable exponent Lebesgue space

$$L^{p(\cdot)}(\Omega) := \left\{u : \Omega \to \mathbb{R} \text{ measurable and } \int_\Omega |u(x)|p(x)dx < \infty\right\},$$

Which is a separable and reflexive Banach space under the Luxemburg norm,

$$\| u \|_{Lp(\cdot)(\Omega)} = \inf\left\{\mu > 0: \int \left|\frac{u(x)}{\mu}\right|^{p(x)} dx \leq 1\right\}.$$

For every $k \in \mathbb{N}$, we define the variable exponent Sobolev space by

$$W^{k,p(\cdot)}(\Omega) := \{u \in L^{p(\cdot)}(\Omega) : D^{\alpha}u \in L^{p(\cdot)}(\Omega), \quad |\alpha| \leq k\},$$

The space $W^{k,p(\cdot)}(\Omega)$ equipped with the norm

$$\| u \|_{W^{k,p(\cdot)}(\Omega)} := \sum_{|\alpha| \leq k} \| D^{\alpha}u \|_{L^{p(\cdot)}(\Omega)},$$

Is a separable and reflexive Banach space, too. Now, we introduce $W_0^{k,p(\cdot)}(\Omega)$ as the closure of $C_0^{\infty}(\Omega)$ in $W^{k,p(\cdot)}(\Omega)$. In the sequel, X will denote the space $W^{2,p(\cdot)}(\Omega) \cap W_0^{1,p(\cdot)}(\Omega)$, which is a reflexive Banach space respect to the norm

$$\begin{aligned} \| u \|_X &:= \| u \|_{W^{2,p(\cdot)}(\Omega)} + \| u \|_{W_0^{1,p(\cdot)}(\Omega)} \\ &= \| u \|_{L^{p(\cdot)}(\Omega)} + \| \nabla u \|_{L^{p(\cdot)}(\Omega)} + \sum_{|\alpha| = k} \| D^{\alpha}u \|_{L^{p(\cdot)}(\Omega)}. \end{aligned}$$

Proposition 1.1. [14] Assume that Ω is a bounded domain with Lipchitz Boundary. The norms $\|\cdot\|_X$ and $\| \nabla u \|_{L^{p(\cdot)}(\Omega)})$ are equivalent on $W^{2,p(\cdot)}(\Omega) \cap W_0^{1,p(\cdot)}(\Omega)$.

$$\|u\| = \inf\left\{\mu > 0: \int_{\Omega} \int \left|\frac{u(x)}{\mu}\right|^{p(x)} dx \leq 1\right\}.$$

Proposition 1.2. [7] Suppose $\frac{1}{p(x)} + \frac{1}{p^*(x)} = 1$, then $L^{p(\cdot)}(\Omega)$ and $L^{p^*(\cdot)}(\Omega)$ are conjugate space, and satisfy the Hölder-type inequality:

$$\left|\int_{\Omega} u(x)v(x)\,dx\right| \leq \left(\frac{1}{p}^{-} + \frac{1}{(p^{*})^{-}}\right)|u|_{p(\cdot)}|v|_{p^{*}(\cdot)} \leq 2|u|_{p(\cdot)}|v|_{p^{*}(\cdot)},$$

Where $p^{-} := \inf_{x\in\bar{\Omega}} p(x)$ and $(p^{*})^{-} := \inf_{x\in\bar{\Omega}} p^{*}(x)$.

Proposition 1.3. [7] Set $\rho(u) := \int_{\Omega} |\Delta u|^{p(x)}\,dx$. For every $u, u_n \in W^{2,p(\cdot)}(\Omega)$, we have

(1) $\| u \| < (=;>)1 \Leftrightarrow \rho(u) < (=;>)1,$

(2) $\min\{\| u \|^{p}; \| u \|^{p^{+}}\} \leq \rho(u) \leq \max\{\| u \|^{p}; \| u \|^{p^{+}}\},$

(3) $\| u_n \| \to 0(\to \infty) \Leftrightarrow \rho(u_n) \to 0(\to \infty):$

Remark 1.1. [11] Let $p \in C_{+}(\bar{\Omega})$ satisfies $p^{-} > \frac{N}{2}$. Then there exist a continuous embedding $X \hookrightarrow W^{2,p^{-}}(\Omega) \cap W_0^{1,p^{-}}(\Omega)$and a compact embedding $W^{2,p^{-}}(\Omega) \cap W_0^{1,p^{-}}(\Omega) \hookrightarrow C^{0}(\bar{\Omega})$, such that X is compactly embedded in $C^{0}(\bar{\Omega})$ and $|u(x)| < K \| u \|$, where K is a positive constant.

Before proving the result, we recall the following multiple critical points theorem of G. Bonanno [3] which can be regarded as supplements of the variational principle of Ricceri [12] which is our main tools.

Theorem 1.1. Let X be a reflexive real Banach space, let $\Phi, \Psi : X \to \mathbb{R}$
Be two Gateaux differentiable functional such that Φ is strongly continuous,
Sequentially weakly lower semi-continuous, and coercive, and Ψ is sequentially weakly upper-semi-continuous. For every $r > \inf_X \Phi$, let

$$\varphi(r) := \inf_{u\in\Phi-1(-\infty,r)} \frac{\left(\sup_{v\in\Phi-1(-\infty,r)}\Psi(v)\right) - \Psi(u)}{r - \Phi(u)},$$
$$\gamma = \lim_{r\to+\infty} \inf \varphi(r), \qquad \delta := \lim_{r\to(\inf_X \Phi)} \inf \varphi(r).$$

Then the following properties hold:

(a) For every $r > \inf_X \Phi$ and every $\lambda \in (0, \frac{1}{\varphi(r)})$ the restriction of the functional $I_\lambda : \Phi - \lambda\Psi$ to $\Phi^{-1}(-\infty,$

r) admits a global minimum, which is a critical point (local minimum) of I_λ in X.

(b) If $\gamma < +\infty$ then, for each $\lambda \in (0, \frac{1}{\gamma})$, the following alternative holds: either

(b1) I_λ possesses a global minimum, or

(b2) there is a sequence $\{u_n\}$ of critical points (local minima) of I_λ

$$\lim_{n\to+\infty} \Phi(u_n) = +\infty.$$

(c) If $\delta < +\infty$ then, for each $\lambda \in (0, \frac{1}{\delta})$, the following alternative holds: either

(c1) there is a global minimum of Φ that is a local minimum of I_λ, or

(c2) there is a sequence $\{u_n\}$ of pairwise distinct critical points (local minima) of $I_\lambda$that weakly converges to a global minimum of Φ with

$$\lim_{n\to+\infty} \Phi(u_n) = \inf_X \Phi$$

Definition 1.1. We say that $u \in X \setminus \{0\}$ is a weak solution of problem (1.1) if $\Delta u = 0$ on $\partial\Omega$ and

$$\int_\Omega a(x, \Delta u)\Delta v dx - \lambda \int_\Omega f(x, u) v dx = 0, \quad \text{for all } v \in X.$$

Define the functionals $\Phi, \Psi : X \to \mathbb{R}$, by

$$\Phi(u) = \int_\Omega A(x; \Delta u) dx \text{ and } \Psi(u) = \int_\Omega F(x; u) dx,$$

and set

$$I_\lambda(u) = \Phi(u) - \lambda\Psi(u), \qquad \text{for all } u \in X.$$

Proposition 1.4. ([4]). The functional $\Phi: X \to \mathbb{R}$ is coercive and mapping $\Phi' : X \to X^*$ is a strictly monotone homeomorphism.

Proof. From Proposition 1.3 and hypothesis (F4) that for $u \in X$ with $\| u \| > 1$,

$$\Phi(u) \geq \int_\Omega \frac{c_2}{p(x)} |\Delta u|^{p(x)} dx \geq \frac{1}{p^+} \rho(u) \geq \frac{1}{p^+} \| u \|^{p^-}$$

So, Φ is coercive. Φ' is strictly monotone see proof in [4].
Furthermore, $\Psi' : X \to X^*$ is compact operator, Indeed, it is enough to show That Ψ' is strongly continuous on X. For this, for fixed $u \in X$, let $u_n \rightharpoonup u$ in X. Remark 1.1 asserts that u_n converges uniformly to u on Ω as $n \to +\infty$. Since f is a Carathéodory function then $f(x, u_n) \to f(x, u)$ strongly as $n \to +\infty$, from Which follows $\Psi'(u_n) \to \Psi'(u)$ strongly as $n \to +\infty$. Then we have that Ψ' is Strongly continuous on X, which implies that Ψ' is a compact operator.

Existence of weak solution

Theorem 2.1. For every λ small enough, i.e.

$$\lambda \in \left(0, \frac{c_2}{p^+ K^{p^-}} \sup_{\gamma>0} \frac{\gamma^{p^-}}{\int_\Omega \sup_{|t|\le\gamma} F(x;\ u)dx}\right),$$

Where K is the constant defined in Remark 1.1, the problem (1.1) admits at least one weak solution $u_\lambda \in X$.

Proof. Our aim is to apply the part (a) of Theorem 1.1 to the problem (1.1).
Let us pick

$$0 < \lambda < \frac{c_2}{p^+ K^{p^-}} \sup_{\gamma>0} \frac{\gamma^{p^-}}{\int_\Omega \sup_{|t|\le\gamma} F(x;\ u)dx}.$$

Hence, there exists $\bar{\gamma} > 0$ such that

$$\lambda p^+ K^{p^-} < c_2 \sup_{\bar{\gamma}>0} \frac{\bar{\gamma}^{p^-}}{\int_\Omega \sup_{|t|\le\bar{\gamma}} F(x;\ u)dx}.$$

Put $r = \frac{c_2}{p^+} (\frac{\bar{\gamma}}{K})^{p^-}$. Moreover, for all $u \in X$ with $\Phi(u) < r$, taking Proposition1.3 into account, we have

$$\| u \| \leq \max\left\{\left(\frac{rp^+}{c_2}\right)^{\frac{1}{p^+}},\ \left(\frac{rp^+}{c_2}\right)^{\frac{1}{p^-}}\right\}.$$

So, from Remark 1.1 for $r > c_2$ we have $|u(x)| \leq \bar{\gamma}$. then

$$\sup_{\Phi(u)\leq r} \Psi(u) = \int_\Omega \sup_{|t|\leq\bar{\gamma}} F(x;\ u)dx.$$

By simple calculations and from the definition of $\varphi(r)$, since $0 \in \Phi^{-1}(-\infty, r)$
And $\Phi(0) = \Psi(0) = 0$, one has

$$\varphi(r) := \inf_{\Phi(u)<r}\left(\frac{\left(\sup_{u'\in\Phi^{-1}(\ ]-\infty,r[\ )}\Psi(u')\right) - \Psi(u)}{r - \Phi(u)}\right) \leq \frac{\sup_{\Phi^{-1}(\ ]-\infty,r[\ )}\Psi}{r}$$

$$\leq \frac{p^+K^{p^-}}{c_2}\frac{\int_\Omega \sup_{|t|\leq\bar{\gamma}} F(x;\ u)dx}{\gamma^{p^-}} \leq \frac{1}{\lambda}.$$

So, since $\lambda \in (0, \frac{1)}{\varphi(r})$, Theorem 1.1 ensures that the functional I_λ admits at least one critical point (local minima).

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