ANALYTICAL SOLUTION OF MULTI-DIMENSIONAL TIME-FRACTIONAL MODEL OF NAVIER-STOKES EQUATION BY RECONSTRUCTION OF VARIATIONAL ITERATION METHOD

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ABSTRACT

In this paper. a new approximate solution of time-fractional order multi-dimensional Navier-Stokes equation is obtained by adopting a semi-analytical scheme "Reconstruction of Variational Iteraton Method (RVIM)". Three test problems are carried out in order to validate and illustrate the efficiency of the method. The scheme is found to be very reliable. effective and efficient powerful technique to solve wide range of problems arising in engineering and sciences. The small size of computation contrary to the other schemes. is its strength.

Keywords: Navier-Stokes equation; Caputo time-fractional. derivative; RVIM; Mittag-Leffler function.

1 Introduction

The idea of fractional derivative was first given by a great mathematician Leibniz. in 1695. in a letter to L'Hospital. Fractional calculus deals with the differential and integral operators with non-integral powers. Noting that the integer-order differential operator is a local operator while the fractionalorder differential operator is non-local. it means that the next state of a system depends not only upon its current state but also upon all of its previous states. It is more realistic and is one of the main reasons why the fractional calculus has become so popular. In the recent years. advances of fractional differential equations have a great attention due to their numerous applications in a wide range of nonlinear complex systems arising in fluid mechanics. viscoelasticity. mathematical biology. life sciences. electrochemistry and physics[1]–[7]For instance. the non-linear oscillation of earthquake can be modeled with fractional derivatives [8]. Fractional differential equations have created attention among the researcher due to exact description of non-linear phenomena. especially in nano-hydrodynamics where continuum assumption does not well. and fractional model can be considered to be a best candidate. These findings invoked the growing interest of studies of the fractal calculus in many branches of science and engineering.

In the recent various analytical techniques such as Homotopy perturbation method (HPM) [9]. homotopy perturbation Sumudu transform method [10], [11]. homotopy analysis method (HAM) [12]. and Adomian decomposition method (ADM) [13], [14].have been developed to solve the fractional partial differential equation. By coupling of HPM and Laplace transform algorithm (LTA). Kumar et al. solved analytically the nonlinear fractional Zakharov Kuznetsov equation in [15]. At first. Keskin and Oturanc [16] introduce reduced differential transform method (RDTM) as a reduced form of differential transform method. and implement it to find the approximate solutions of partial (and factional partial) differential equations [16]. [17]. Fractional reduced differential transform method (FRDTM) has been adopted in many articles to solve the differential equations prevailing in mathematics. physics and engineering [18], [19], [20], [21], [22], [23], [24], [25], [26],[27],[28], [29].

A famous governing equation of motion of viscus fluid flow called Navier-Stokes (NS) equation has been derived in 1822 [30]. The equation can be regarded as Newton's second law of motion for fluid substances. and is a combination of Momentum equation. continuity equation and the energy equation. This equation describes many physical things such as ocean currents. liquid flow in pipes. blood flow and air flow around the wings of an aircraft. The fractional modeling of NS equations was first done in 2005 by El-Shahed and Salem [31] . The authors [31] generalized the classical NS equations using Laplace transform. finite Hankel transforms and finite Fourier Sine transform. By coupling of HPM and LTA. Kumar et al [32]. solved analytically a nonlinear fractional model of NS equation. Ragab et al [12]. and Ganji et al[12] solved nonlinear time-fractional NS equation by adopting HAM. Birajdar [14] and Momani and Odibat [15] adopted ADM for numerical computation of time-fractional NS equation. Analytical solution of time-fractional NS equation is obtained using coupling of ADM and LTA by Kumar et al [15]. while Chaurasia and Kumar [33] solved the same equation by coupling of Laplace transform and finite Hankel transform. This paper presents an approximate analytic solution of Reconstruction of Variational Iteration Method. of NS equation by adopting (RVIM).

2 Basic Definition

Definition 2.1 : The Riemann-Liouville fractional integral of $f(t)$ of the order $\alpha \geq 0$ is defined as

$$J_t^{\alpha} f(t) = \begin{cases} f(t) & if\ \alpha = 0. \\ \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) \mathrm{d}\tau & if\ \alpha > 0. \end{cases} \quad (1)$$

where Γ denotes gamma function

$$\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1} \mathrm{d}t.\ z \in \mathbb{C}.$$

Definition 2.2: The fractional derivative of f of the order $\alpha \geq 0.$ in Caputo sense is defined as

$$D_t^{\alpha} f(t) = J_t^{m-1} D_t^m f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} f^{(m)}(\tau) \mathrm{d}\tau. \quad (2)$$

for $m-1 < \alpha \leq m.\ m \in \mathbb{N}\ .\ t > 0.\ f \in \mathbb{C}^m.$

The basic properties of Caputo fractional derivative are given as follows:

Definition 2.3: The Mittag-Leffler functions. which is generalization of the exponential function. is defined as:

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+1)}. \quad E_{\alpha.\beta}(z) = \sum_{k=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+\beta)} \quad (3)$$

Lemma 2.1: Let $m-1 < \alpha \geq m$ and $f \in \mathbb{C}^m.$ then

$$D_t^{\alpha} J_t^{\alpha} f(t) = f(t).$$

$$J_t^{\alpha} D_t^{\alpha} f(t) = f(t) - \sum_{k=0}^{m} f^k(0^+) \frac{t^k}{k!}.\quad t > 0.$$

3 Implementation of RVIM on Navier-Stokes equation

In this section we introduce an approximate analytical method to solve Navier-Stokes system of fractional differential equation of order $0 < \alpha < 1$. Hesameddini and Latifzadeh [34] presented the Reconstruction of Variational Iteration Method (RVIM) for differential equations of integer and fractional order. Here. we expand this approach to solve multi-dimensional. time-fractional model of Navier-Stokes equation. In Cartesian co-ordinates. the following equations becomes.

$$\begin{cases} D_t^\alpha u + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = \rho_0\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) - \frac{1}{\rho}\frac{\partial p}{\partial x} \\ D_t^\alpha v + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = \rho_0\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) - \frac{1}{\rho}\frac{\partial p}{\partial y} \\ D_t^\alpha w + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = \rho_0\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) - \frac{1}{\rho}\frac{\partial p}{\partial z} \end{cases} \quad (4)$$

with initial conditions $u_0(x.y.z.t) = f(x.y.z.0).v_0(x.y.z.t) = g(x.y.z.0).w_0(x.y.z.t) = h(x.y.z.0)$.

In (4) where the operator D_t^α is the Caputo fractional derivatives and $m-1 < \alpha \leq m$ and $\rho_0 = \frac{\eta}{\rho}$ denotes the kinematic viscosity of the flow. which η denotes dynamic viscosity and ρ is density. if p is known. then $g_1 = \frac{1}{\rho}\frac{\partial p}{\partial x}.g_2 = \frac{1}{\rho}\frac{\partial p}{\partial y}.\ g_3 = \frac{1}{\rho}\frac{\partial p}{\partial z}$ and $\nabla^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)$. By taking Laplas Transform from both side of equation (4). with respect to the independent variable t and using the homogeneous initial condition. we get

$$\begin{aligned} s^\alpha\mathcal{L}\{u(x.y.z.t)\} - s^{\alpha-1}u(x.y.z.0) &= \mathcal{L}\left\{-u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} - w\frac{\partial u}{\partial z} + \rho_0\nabla^2 u - g_1\right\}. \\ s^\alpha\mathcal{L}\{v(x.y.z.t)\} - s^{\alpha-1}v(x.y.z.0) &= \mathcal{L}\left\{-u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} - w\frac{\partial u}{\partial z} + \rho_0\nabla^2 v - g_2\right\}. \\ s^\alpha\mathcal{L}\{w(x.y.z.t)\} - s^{\alpha-1}w(x.y.z.0) &= \mathcal{L}\left\{-u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} - w\frac{\partial u}{\partial z} + \rho_0\nabla^2 w - g_3\right\}. \end{aligned} \quad (5)$$

Therefore

$$\begin{aligned} \mathcal{L}\{u(x.y.z.t)\} &= \frac{1}{s}u(x.y.z.0) + \frac{1}{s^\alpha}\mathcal{L}\left\{-u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} - w\frac{\partial u}{\partial z} + \rho_0\nabla^2 u - g_1\right\}. \\ \mathcal{L}\{v(x.y.z.t)\} &= \frac{1}{s}v(x.y.z.0) + \frac{1}{s^\alpha}\mathcal{L}\left\{-u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} - w\frac{\partial u}{\partial z} + \rho_0\nabla^2 v - g_2\right\}. \\ \mathcal{L}\{w(x.y.z.t)\} &= \frac{1}{s}w(x.y.z.0) + \frac{1}{s^\alpha}\mathcal{L}\left\{-u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} - w\frac{\partial u}{\partial z} + \rho_0\nabla^2 w - g_3\right\}. \end{aligned} \quad (6)$$

suppose $-u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} - w\frac{\partial u}{\partial z} + \rho_0\nabla^2 u - g_1 = f(t.x.y.z.\frac{\partial u}{\partial x}.\frac{\partial u}{\partial y}.\frac{\partial u}{\partial z}.\frac{\partial u^2}{\partial x^2}.\frac{\partial u^2}{\partial y^2}.\frac{\partial u^2}{\partial z^2})$. $-u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} - w\frac{\partial u}{\partial z} + \rho_0\nabla^2 v - g_2 = g(t.x.y.z.\frac{\partial u}{\partial x}.\frac{\partial u}{\partial y}.\frac{\partial u}{\partial z}.\frac{\partial u^2}{\partial x^2}.\frac{\partial u^2}{\partial y^2}.\frac{\partial u^2}{\partial z^2})$. $-u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} - w\frac{\partial u}{\partial z} + \rho_0\nabla^2 w - g_3 = h(t.x.y.z.\frac{\partial u}{\partial x}.\frac{\partial u}{\partial y}.\frac{\partial u}{\partial z}.\frac{\partial u^2}{\partial x^2}.\frac{\partial u^2}{\partial y^2}.\frac{\partial u^2}{\partial z^2})$. Now by applying the inverse Laplace transform to both side of equation(7). and using the convolution theorem we

$$\begin{aligned}
u(x.y.z.t) &= f_0(x.y.z.t) + \mathcal{L}^{-1}\left\{\frac{1}{s^\alpha}F\left(s.x.y.z.\frac{\partial u}{\partial x}.\frac{\partial u}{\partial y}.\frac{\partial u}{\partial z}.\frac{\partial u^2}{\partial x^2}.\frac{\partial u^2}{\partial y^2}.\frac{\partial u^2}{\partial z^2}\right)\right\} \\
&= f_0(x.y.z.t) + \frac{t^{\alpha-1}}{\Gamma(\alpha)} \star f\left(t.x.y.z.\frac{\partial u}{\partial x}.\frac{\partial u}{\partial y}.\frac{\partial u}{\partial z}.\frac{\partial u^2}{\partial x^2}.\frac{\partial u^2}{\partial y^2}.\frac{\partial u^2}{\partial z^2}\right) \\
&= f_0(x.y.z.t) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f\left(t.x.y.z.\frac{\partial u}{\partial x}.\frac{\partial u}{\partial y}.\frac{\partial u}{\partial z}.\frac{\partial u^2}{\partial x^2}.\frac{\partial u^2}{\partial y^2}.\frac{\partial u^2}{\partial z^2}\right) \mathrm{d}\tau. \\
v(x.y.z.t) &= g_0(x.y.z.t) + \mathcal{L}^{-1}\left\{\frac{1}{s^\alpha}G\left(s.x.y.z.\frac{\partial u}{\partial x}.\frac{\partial u}{\partial y}.\frac{\partial u}{\partial z}.\frac{\partial u^2}{\partial x^2}.\frac{\partial u^2}{\partial y^2}.\frac{\partial u^2}{\partial z^2}\right)\right\} \\
&= g_0(x.y.z.t) + \frac{t^{\alpha-1}}{\Gamma(\alpha)} \star g\left(t.x.y.z.\frac{\partial u}{\partial x}.\frac{\partial u}{\partial y}.\frac{\partial u}{\partial z}.\frac{\partial u^2}{\partial x^2}.\frac{\partial u^2}{\partial y^2}.\frac{\partial u^2}{\partial z^2}\right) \qquad (7) \\
&= g_0(x.y.z.t) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} g\left(t.x.y.z.\frac{\partial u}{\partial x}.\frac{\partial u}{\partial y}.\frac{\partial u}{\partial z}.\frac{\partial u^2}{\partial x^2}.\frac{\partial u^2}{\partial y^2}.\frac{\partial u^2}{\partial z^2}\right) \mathrm{d}\tau. \\
w(x.y.z.t) &= h_0(x.y.z.t) + \mathcal{L}^{-1}\left\{\frac{1}{s^\alpha}H\left(s.x.y.z.\frac{\partial u}{\partial x}.\frac{\partial u}{\partial y}.\frac{\partial u}{\partial z}.\frac{\partial u^2}{\partial x^2}.\frac{\partial u^2}{\partial y^2}.\frac{\partial u^2}{\partial z^2}\right)\right\} \\
&= h_0(x.y.z.t) + \frac{t^{\alpha-1}}{\Gamma(\alpha)} \star h\left(t.x.y.z.\frac{\partial u}{\partial x}.\frac{\partial u}{\partial y}.\frac{\partial u}{\partial z}.\frac{\partial u^2}{\partial x^2}.\frac{\partial u^2}{\partial y^2}.\frac{\partial u^2}{\partial z^2}\right) \\
&= h_0(x.y.z.t) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} h\left(t.x.y.z.\frac{\partial u}{\partial x}.\frac{\partial u}{\partial y}.\frac{\partial u}{\partial z}.\frac{\partial u^2}{\partial x^2}.\frac{\partial u^2}{\partial y^2}.\frac{\partial u^2}{\partial z^2}\right) \mathrm{d}\tau.
\end{aligned}$$

according to [34] by imposing to initial condition to obtain the solution of equation (4) . we construct an iteration formula as follows:

$$\begin{aligned}
&u_{n+1}(x.y.z.t) = f_0(x.y.z.t) \\
&+\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f\left(t.x.y.z.\frac{\partial u_n}{\partial x}.\frac{\partial u_n}{\partial y}.\frac{\partial u_n}{\partial z}.\frac{\partial u_n^2}{\partial x^2}.\frac{\partial u_n^2}{\partial y^2}.\frac{\partial u_n^2}{\partial z^2}\right) \mathrm{d}\tau. \\
&v_{n+1}(x.y.z.t) = g_0(x.y.z.t) \\
&+\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} g\left(t.x.y.z.\frac{\partial u_n}{\partial x}.\frac{\partial u_n}{\partial y}.\frac{\partial u_n}{\partial z}.\frac{\partial u_n^2}{\partial x^2}.\frac{\partial u_n^2}{\partial y^2}.\frac{\partial u_n^2}{\partial z^2}\right) \mathrm{d}\tau. \qquad (8) \\
&w_{n+1}(x.y.z.t) = h_0(x.y.z.t) \\
&+\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} h\left(t.x.y.z.\frac{\partial u_n}{\partial x}.\frac{\partial u_n}{\partial y}.\frac{\partial u_n}{\partial z}.\frac{\partial u_n^2}{\partial x^2}.\frac{\partial u_n^2}{\partial y^2}.\frac{\partial u_n^2}{\partial z^2}\right) \mathrm{d}\tau.
\end{aligned}$$

where $f_0(x.y.z.t).g_0(x.y.z.t)).h_0(x.y.z.t).$ is initial solution. By the above iteration each term will be determined by the previous term in the approximation of iteration formula can be entirely evaluated. Consequently the solution may be written as:

$$\begin{aligned}
u(x.y.z.t) &= \lim_{\to\infty} u_n(x.y.z.t). \\
v(x.y.z.t) &= \lim_{\to\infty} v_n(x.y.z.t). \qquad (9) \\
w(x.y.z.t) &= \lim_{\to\infty} w_n(x.y.z.t).
\end{aligned}$$

3.1 Illustrative examples

Consider time-fractional order 2-dimensional NS equation with $g_1 = -g_2 = g$ as

$$\begin{aligned}
D_t^\alpha u + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= \rho_0\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + g. \\
D_t^\alpha v + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} &= \rho_0\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) - g. \qquad (10)
\end{aligned}$$

with the initial conditions

$$u(x.y.0) = -\sin(x+y). \quad v(x.y.0) = \sin(x+y)$$

Using RVIM on the above two equations. we obtained the following recurrence relation:

$$\begin{aligned} &u_{n+1}(x.y.t) = f_0(x.y.t) \\ &+\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left[-u_n\frac{\partial u_n}{\partial x} - v_n\frac{\partial u_n}{\partial y} + \rho_0\left(\frac{\partial^2 u_n}{\partial x^2} + \frac{\partial^2 u_n}{\partial y^2}\right) + g\right]\mathrm{d}\tau \\ &v_{n+1}(x.y.t) = g_0(x.y.t) \\ &+\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left[-u_n\frac{\partial v_n}{\partial x} - v_n\frac{\partial v_n}{\partial y} + \rho_0\left(\frac{\partial^2 v_n}{\partial x^2} + \frac{\partial^2 v_n}{\partial y^2}\right) - g\right]\mathrm{d}\tau \end{aligned} \tag{11}$$

for $n = 0$ we obtain as:

$$\begin{aligned} &u_1(x.y.t) = f_0(x.y.t) \\ &+\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left[-u_0\frac{\partial u_0}{\partial x} - v_0\frac{\partial u_0}{\partial y} + \rho_0\left(\frac{\partial^2 u_0}{\partial x^2} + \frac{\partial^2 u_0}{\partial y^2}\right) + g\right]\mathrm{d}\tau \\ &v_1(x.y.t) = g_0(x.y.t) \\ &+\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left[-u_0\frac{\partial v_0}{\partial x} - v_0\frac{\partial v_0}{\partial y} + \rho_0\left(\frac{\partial^2 v_0}{\partial x^2} + \frac{\partial^2 v_0}{\partial y^2}\right) - g\right]\mathrm{d}\tau \end{aligned} \tag{12}$$

by simplifying we obtain that:

$$\begin{aligned} u_1(x.y.t) = \; & -\sin(x+y) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}(2\rho_0\sin(x+y) + g)\mathrm{d}\tau \\ & = \sin(x+y)\left(-1 + \frac{2\rho_0 t^\alpha}{\Gamma(1+\alpha)}\right) + g\frac{t^\alpha}{\Gamma(1+\alpha)} \\ v_1(x.y.t) = \; & \sin(x+y) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}(2\rho_0\sin(x+y) - g)\mathrm{d}\tau \\ & = \sin(x+y)\left(1 - \frac{2\rho_0 t^\alpha}{\Gamma(1+\alpha)}\right) - g\frac{t^\alpha}{\Gamma(1+\alpha)} \end{aligned} \tag{13}$$

for $n = 1$ we obtain as:

$$\begin{aligned} u_2(x.y.t) &= \sin(x+y)\left(-1 + \frac{2\rho_0 t^\alpha}{\Gamma(1+\alpha)} - \frac{(2\rho_0)^2 t^{2\alpha}}{\Gamma(1+2\alpha)}\right) + g\frac{t^\alpha}{\Gamma(1+\alpha)} \\ v_2(x.y.t) &= \sin(x+y)\left(1 - \frac{2\rho_0 t^\alpha}{\Gamma(1+\alpha)} + \frac{(2\rho_0)^2 t^{2\alpha}}{\Gamma(1+2\alpha)}\right) - g\frac{t^\alpha}{\Gamma(1+\alpha)} \end{aligned} \tag{14}$$

recently we get:

$$\begin{aligned} u(x.y.t) \; &= \lim_{n\to\infty} u_n(x.y.t) = -\sin(x+y)\sum_{k=0}^{\infty}\frac{(-2\rho_0 t^\alpha)^k}{\Gamma(1+k\alpha)} + \frac{gt^\alpha}{\Gamma(1+\alpha)} \\ & -\sin(x+y)E_\alpha(-2\rho_0 t^\alpha) - \frac{gt^\alpha}{\Gamma(\alpha+1)}. \\ v(x.y.t) \; &= \lim_{n\to\infty} v_n(x.y.t) = \sin(x+y)\sum_{k=0}^{\infty}\frac{(-2\rho_0 t^\alpha)^k}{\Gamma(1+k\alpha)} - \frac{gt^\alpha}{\Gamma(1+\alpha)} \\ & \sin(x+y)E_\alpha(-2\rho_0 t^\alpha) - \frac{gt^\alpha}{\Gamma(\alpha+1)}. \end{aligned} \tag{15}$$

for $g = 0$ and $\alpha = 1$ we obtain as

$$\begin{aligned} u(x.y.t) &= -e^{-2\rho_0 t}\sin(x+y.) \\ v(x.y.t) &= e^{-2\rho_0 t}\sin(x+y). \end{aligned} \tag{16}$$

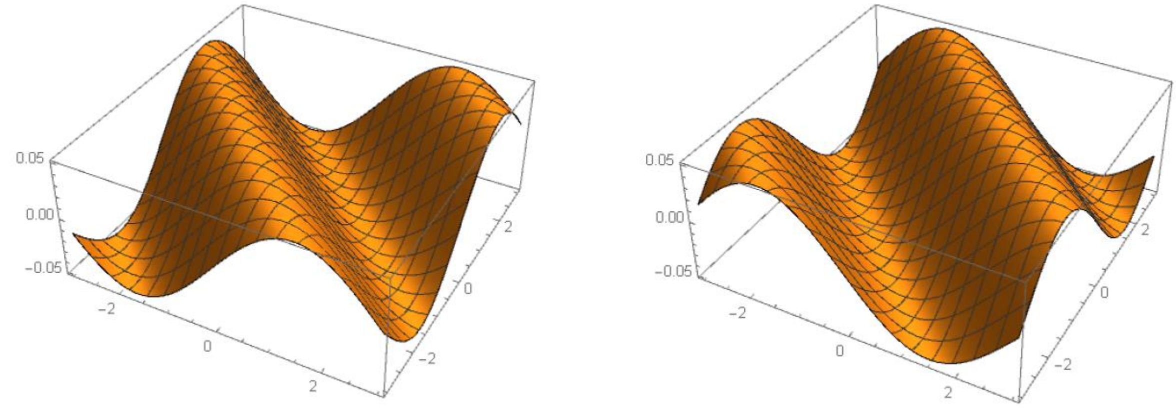


Figure 1: The behavior of u and v of NS equation in 2.1 at $t = 3$ with parameters $\alpha = 1. g = 0. \rho_0 = 0.5$

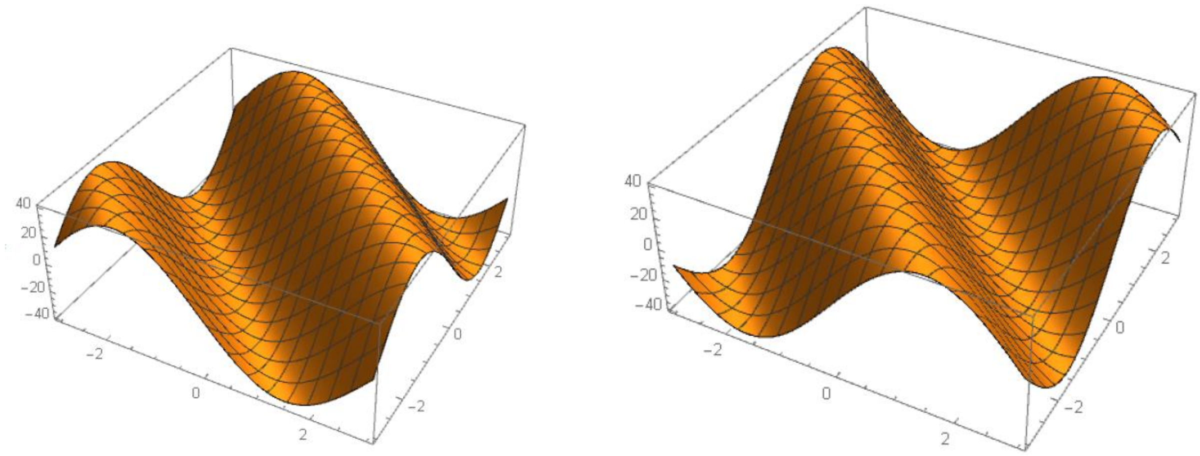
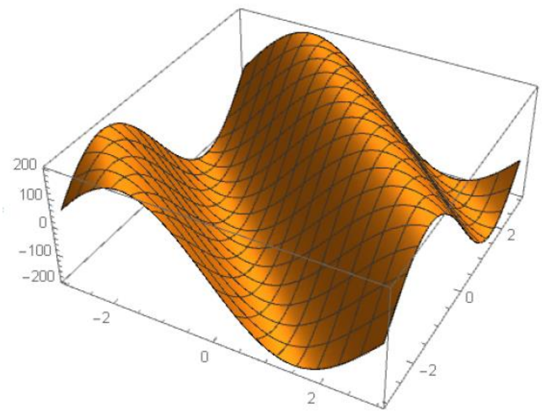


Figure 2: The behavior of u and v of NS equation in 2.1 at $t = 3$ with parameters $\alpha = 0.5. g = 0. \rho_0 = 0.5$



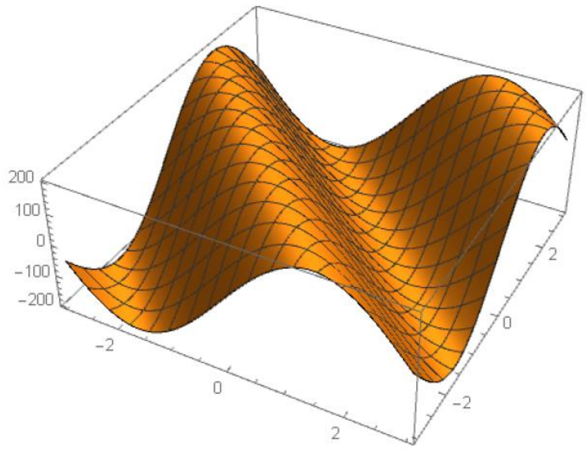


Figure 3: The behavior of u and v of NS equation in 2.1 at $t = 3$ with parameters $\alpha = 0.1. g = 0. \rho_0 = 0.5$

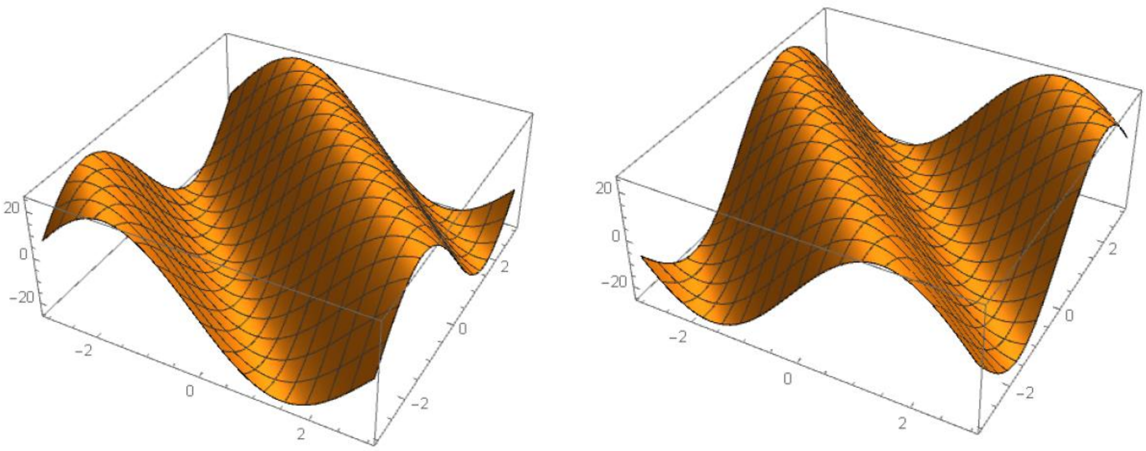


Figure 4: The behavior of u and v of NS equation in 2.1 at $t = 3$ with parameters $\alpha = 0.8. g = 0. \rho_0 = 0.5$

Consider time-fractional order two dimensional (10) subject to the initial condition:

$$u(x.y.0) = -e^{x+y}. \quad v(x.y.0) = e^{x+y} \tag{17}$$

for $n = 0$ we obtain as:

$$\begin{aligned} u_1(x.y.t) &= f_0(x.y.t) \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \left[-u_0 \frac{\partial u_0}{\partial x} - v_0 \frac{\partial u_0}{\partial y} + \rho_0 \left(\frac{\partial^2 u_0}{\partial x^2} + \frac{\partial^2 u_0}{\partial y^2} \right) + g \right] \mathrm{d}\tau \\ v_1(x.y.t) &= g_0(x.y.t) \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \left[-u_0 \frac{\partial v_0}{\partial x} - v_0 \frac{\partial v_0}{\partial y} + \rho_0 \left(\frac{\partial^2 v_0}{\partial x^2} + \frac{\partial^2 v_0}{\partial y^2} \right) - g \right] \mathrm{d}\tau \end{aligned} \tag{18}$$

by simplifying we obtain that:

$$\begin{aligned} u_1(x.y.t) = & -e^{x+y}+\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}(2\rho_0 e^{x+y}+g)\mathrm{d}\tau \\ & = -e^{x+y}\left(1+\frac{2\rho_0 t^\alpha}{\Gamma(1+\alpha)}\right)+g\frac{t^\alpha}{\Gamma(1+\alpha)} \\ v_1(x.y.t) = & \ e^{x+y}+\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}(2\rho_0 e^{x+y}-g)\mathrm{d}\tau \\ & = e^{x+y}\left(1+\frac{2\rho_0 t^\alpha}{\Gamma(1+\alpha)}\right)-g\frac{t^\alpha}{\Gamma(1+\alpha)} \end{aligned} \quad (19)$$

for $n=1$ we obtain as:

$$\begin{aligned} u_2(x.y.t) &= -e^{x+y}\left(1+\frac{2\rho_0 t^\alpha}{\Gamma(1+\alpha)}+\frac{(2\rho_0)^2 t^{2\alpha}}{\Gamma(1+2\alpha)}\right)+g\frac{t^\alpha}{\Gamma(1+\alpha)} \\ v_2(x.y.t) &= e^{x+y}\left(1+\frac{2\rho_0 t^\alpha}{\Gamma(1+\alpha)}+\frac{(2\rho_0)^2 t^{2\alpha}}{\Gamma(1+2\alpha)}\right)-g\frac{t^\alpha}{\Gamma(1+\alpha)} \end{aligned} \quad (20)$$

recently we get:

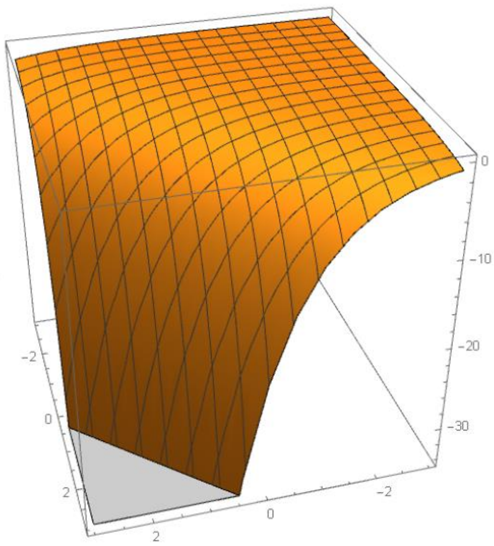
$$\begin{aligned} u(x.y.t) &= \lim_{n\to\infty} u_n(x.y.t) = -e^{x+y}\sum_{k=0}^{\infty}\frac{(2\rho_0 t^\alpha)^k}{\Gamma(1+k\alpha)}+\frac{gt^\alpha}{\Gamma(1+\alpha)} \\ &= -e^{x+y}E_\alpha(2\rho_0 t^\alpha)-\frac{gt^\alpha}{\Gamma(\alpha+1)}. \\ v(x.y.t) &= \lim_{n\to\infty} v_n(x.y.t) = e^{x+y})\sum_{k=0}^{\infty}\frac{(2\rho_0 t^\alpha)^k}{\Gamma(1+k\alpha)}-\frac{gt^\alpha}{\Gamma(1+\alpha)} \\ &= e^{x+y}E_\alpha(2\rho_0 t^\alpha)-\frac{gt^\alpha}{\Gamma(\alpha+1)}. \end{aligned} \quad (21)$$

for $g=0$ and $\alpha=1$ we obtain as

$$\begin{aligned} u(x.y.t) &= -e^{-2\rho_0 t}\sin(x+y.) \\ v(x.y.t) &= e^{-2\rho_0 t}\sin(x+y). \end{aligned} \quad (22)$$

for $g=0$ and $\alpha=1$ we obtain as:

$$u(x.y.t) = -e^{x+y+2\rho_0 t}. \quad v(x.y.t) = e^{x+y+2\rho_0 t}. \quad (23)$$



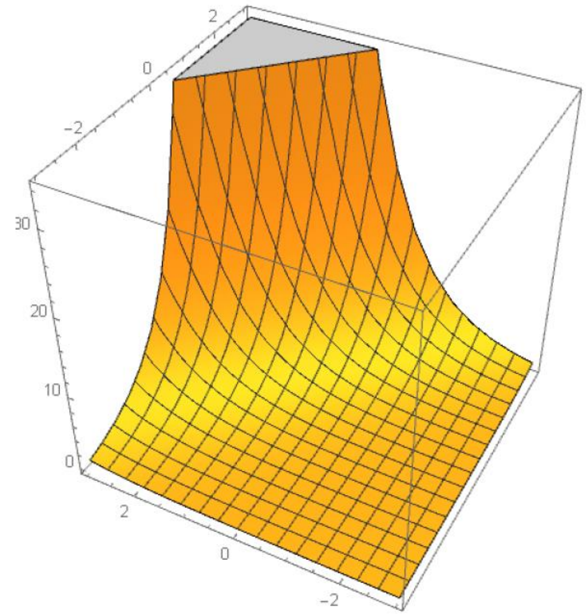


Figure 5: The behavior of u and v of NS equation in 4 at $t = 0.05$ with parameters $\alpha = 1. g = 0. \rho_0 = 0.$

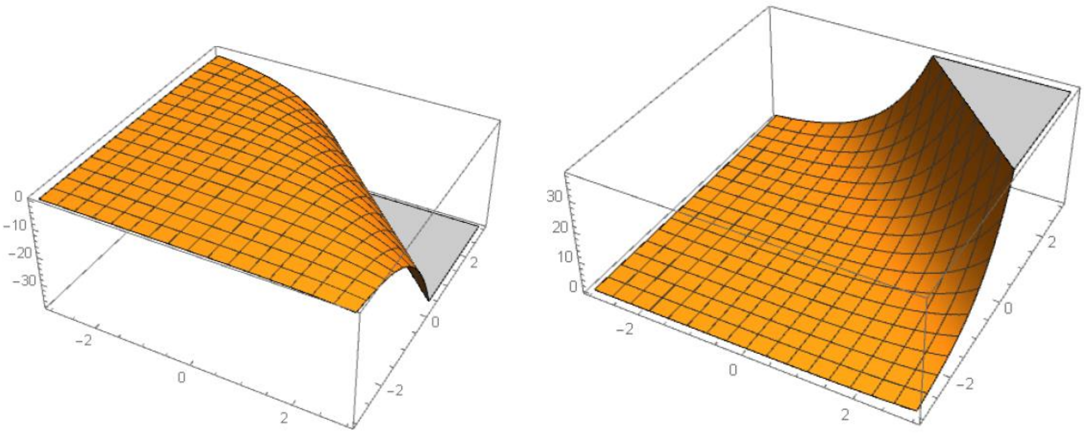


Figure 6: The behavior of u and v of NS equation in 4 at $t = 0.05$ with parameters $\alpha = 0.5. g = 0. \rho_0 = 0.5$

Consider time-fractional order three dimensional (4) with $g_1 = g_2 = g_3 = 0.$ subject to the initial condition:

$$u(x.y.z.0) = -0.5x + y + z. \quad v(x.y.z.0) = x - 0.5y + z.$$
$$w(x.y.z.0) = x + y - 0.5z. \qquad (24)$$

Using RVIM on the above two equations. we obtained the following recurrence relation:

$$
\begin{aligned}
&u_{n+1}(x.y.t) = f_0(x.y.t)\\
&+\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left[-u_n\frac{\partial u_n}{\partial x}-v_n\frac{\partial u_n}{\partial y}-w_n\frac{\partial u_n}{\partial z}+\rho_0\left(\frac{\partial^2 u_n}{\partial x^2}+\frac{\partial^2 u_n}{\partial y^2}+\frac{\partial^2 u_n}{\partial z^n}\right)\right]\mathrm{d}\tau\\
&v_{n+1}(x.y.t) = g_0(x.y.t)\\
&+\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left[-u_n\frac{\partial v_n}{\partial x}-v_n\frac{\partial v_n}{\partial y}-w_n\frac{\partial v_n}{\partial z}+\rho_0\left(\frac{\partial^2 v_n}{\partial x^2}+\frac{\partial^2 v_n}{\partial y^2}+\frac{\partial^2 v_n}{\partial z^n}\right)\right]\mathrm{d}\tau\\
&w_{n+1}(x.y.t) = g_0(x.y.t)\\
&+\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left[-u_n\frac{\partial w_n}{\partial x}-v_n\frac{\partial w_n}{\partial y}-w_n\frac{\partial w_n}{\partial z}+\rho_0\left(\frac{\partial^2 w_n}{\partial x^2}+\frac{\partial^2 w_n}{\partial y^2}+\frac{\partial^2 w_n}{\partial z^n}\right)\right]\mathrm{d}\tau
\end{aligned}
\tag{25}
$$

for $n = 0$ we obtain as:

$$
\begin{aligned}
&u_1(x.y.t) = f_0(x.y.t)\\
&+\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left[-u_0\frac{\partial u_0}{\partial x}-v_0\frac{\partial u_0}{\partial y}-w_0\frac{\partial u_0}{\partial z}+\rho_0\left(\frac{\partial^2 u_0}{\partial x^2}+\frac{\partial^2 u_0}{\partial y^2}+\frac{\partial^2 u_0}{\partial z^0}\right)\right]\mathrm{d}\tau\\
&v_1(x.y.t) = g_0(x.y.t)\\
&+\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left[-u_0\frac{\partial v_0}{\partial x}-v_0\frac{\partial v_0}{\partial y}-w_0\frac{\partial v_0}{\partial z}+\rho_0\left(\frac{\partial^2 v_0}{\partial x^2}+\frac{\partial^2 v_0}{\partial y^2}+\frac{\partial^2 v_0}{\partial z^0}\right)\right]\mathrm{d}\tau\\
&w_{n+1}(x.y.t) = h_0(x.y.t)\\
&+\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\left[-u_0\frac{\partial w_0}{\partial x}-v_0\frac{\partial w_0}{\partial y}-w_0\frac{\partial w_0}{\partial z}+\rho_0\left(\frac{\partial^2 w_0}{\partial x^2}+\frac{\partial^2 w_0}{\partial y^2}+\frac{\partial^2 w_0}{\partial z^2}\right)\right]\mathrm{d}\tau
\end{aligned}
\tag{26}
$$

by simplifying we obtain that:

$$
\begin{aligned}
&u_1(x.y.t) = -0.5x+y+z+\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}(-2.25x)\mathrm{d}\tau = -0.5x+y+z-\frac{2.25xt^\alpha}{\Gamma(1+\alpha)}\\
&v_1(x.y.t) = x-0.5y+z+\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}(-2.25y)\mathrm{d}\tau = x-0.5y+z-\frac{2.25yt^\alpha}{\Gamma(1+\alpha)}\\
&w_1(x.y.t) = x+y-0.5z+\frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}(-2.25z)\mathrm{d}\tau = x+y-0.5z-\frac{2.25zt^\alpha}{\Gamma(1+\alpha)}
\end{aligned}
$$

recently we get:

$$
\begin{aligned}
u(x.y.z.t) &= \lim_{n\to\infty} u_n(x.y.z.t) = -0.5x+y+z-\frac{2.25}{\Gamma(1+\alpha)}xt^\alpha+\frac{2(2.25)}{\Gamma(1+2\alpha)}(-0.5x+y+z)t^{2\alpha}\\
&-\frac{(2.25)^2}{\Gamma(1+3\alpha)}\left(4+\frac{1+2\alpha}{(\Gamma(1+\alpha))^2}\right)xt^{3\alpha}\\
&+\frac{(2.25)^2}{\Gamma(1+4\alpha)}\left(8+\frac{2\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2}+\frac{4(\Gamma(1+3\alpha))}{\Gamma(1+\alpha)\Gamma(1+2\alpha)}\right)(-0.5x+y+z)t^{4\alpha}+\\
v(x.y.z.t) &= \lim_{n\to\infty} v_n(x.y.z.t) = x-0.5y+z-\frac{2.25}{\Gamma(1+\alpha)}yt^\alpha+\frac{2(2.25)}{\Gamma(1+2\alpha)}(x-0.5y+z)t^{2\alpha}\\
&-\frac{(2.25)^2}{\Gamma(1+3\alpha)}\left(4+\frac{1+2\alpha}{(\Gamma(1+\alpha))^2}\right)yt^{3\alpha}\\
&+\frac{(2.25)^2}{\Gamma(1+4\alpha)}\left(8+\frac{2\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2}+\frac{4(\Gamma(1+3\alpha))}{\Gamma(1+\alpha)\Gamma(1+2\alpha)}\right)(x-0.5y+z)t^{4\alpha}+\cdots\\
w(x.y.z.t) &= \lim_{n\to\infty} w_n(x.y.z.t) = x+y-0.5z-\frac{2.25}{\Gamma(1+\alpha)}zt^\alpha+\frac{2(2.25)}{\Gamma(1+2\alpha)}(x+y-0.5z)t^{2\alpha}\\
&-\frac{(2.25)^2}{\Gamma(1+3\alpha)}\left(4+\frac{1+2\alpha}{(\Gamma(1+\alpha))^2}\right)zt^{3\alpha}\\
&+\frac{(2.25)^2}{\Gamma(1+4\alpha)}\left(8+\frac{2\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2}+\frac{4(\Gamma(1+3\alpha))}{\Gamma(1+\alpha)\Gamma(1+2\alpha)}\right)(x+y-0.5z)t^{4\alpha}+\cdots
\end{aligned}
$$

which is the required exact solution. For $\alpha = 1.$ we have:

$$\begin{cases} u(x.y.z.t) & = (-0.5x + y + z)(1 + 2.25t^2 + (2.25)^2t^4 + \cdots) - 2.25xt(1 + 2.25t^2 + \cdots) \\ & = \frac{-0.5x+y+z-2.25xt}{1-2.25t^2}. \\ v(x.y.z.t) & = (x - 0.5y + z)(1 + 2.25t^2 + (2.25)^2t^4 + \cdots) - 2.25yt(1 + 2.25t^2 + \cdots) \\ & = \frac{x-0.5y+z-2.25yt}{1-2.25t^2}. \\ v(x.y.z.t) & = (x + y - 0.5z)(1 + 2.25t^2 + (2.25)^2t^4 + \cdots) - 2.25zt(1 + 2.25t^2 + \cdots) \\ & = \frac{x+y-0.5z-2.25zt}{1-2.25t^2}. \end{cases}$$

(27)

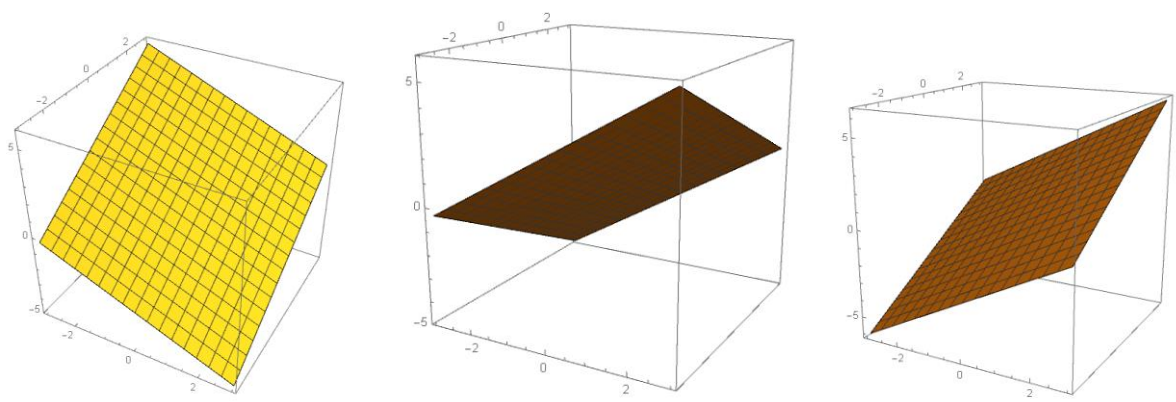


Figure 7: The velocity profile $(u.v.w)$ of NS equation in 6 at $t = 0.1.$ $z = 0.5$ with $\alpha = 1.$

4 Conclusion

In this paper. Reconstruction Variational Iteration method is adopted for the numerical simulation of time-fractional model of Navier-Stokes equations with initial conditions. The fractional derivative is considered in the Caputo sense. The analytical results have been given in terms of a power series. Three test problems are carried out in order to validate and illustrate the efficiency of the method. The proposed solutions agree excellently with HPM [13] and ADM [14]. and are approximated without any discretization. transformation. perturbation. or restrictive conditions. However. the performed calculations show that the described method needs very small size of computation in comparison with HPM [13] and ADM [14]. Small size of computation contrary to the other schemes. is the strength of the scheme.

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