THE AXIOM OF CHOICE EQUIVALENTS AND ITS APPLICATIONS

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ABSTRACT

In this article, we examine the Axiom of Choice and other equivalent principles. We will discuss the fact that many mathematical conclusions, which were assumed to be clear for years, can be expressed based on the Axiom of choice. Choosing ten numbers (they don't have to be unequal) from ten boxes is simple and basic. This choice may not be obvious when the number of choices is infinite. All the above concepts revolve around the principle called the Axiom of Choice. In this article, we present the principles equivalent to the Axiom of Choice with a detailed proof, and we also prove some important theorems related to other parts of mathematics with the help of the Axiom of Choice and its equivalents.

Keywords: Axiom of Choice, Tukey's Lemma, Principle, Zorn's Lemma, well-Ordered Theorem.

INTRODUCTION

The Axiom of Choice is one of the basic and important principles that a large number of mathematical results can be expressed based on it. To express the problem precisely, suppose that non-empty sets $A_1, A_2, \ldots., and\ A_n$ re assumed. We want to find the set such that $A_1 \cap A \neq \emptyset$,$A_2 \cap A \neq \emptyset$,... ,and $A_n \cap A \neq \emptyset$. This can be done in different ways. Now we present a method. Since $A_1 \neq \emptyset$ we choose $a_1 \in A_1$as optional. If $a_1 \in A_2$ then $a_2 = a_1$. If $a_1 \notin A_2$, then $a_2 \in A_2$ is selected as before. We repeat this method by induction and consider the set A as $\{a_1, a_2, \ldots, a_n\}$. It is clear that A has the required conditions.

In the second method, we choose $a_i \in A_i$ arbitrarily for $i = 1,2, \ldots, n$. In the first method, a smaller set may be obtained. Now suppose that the non-empty sequence of sets $\{A_n\}_{n \in N}$ are given and A is the set with the previous conditions. There is no problem here either, it is enough to select $a_1 \in A_1$first. In the second step, we select $a_2 \in A_2$ and continue this work. It seems that set is obtained. The only problem is that the number of operations is infinite. Now suppose that the non-empty family of non-empty sets A_q is given for every $q \in \mathbb{Q}$. In this case, the previous method cannot be used, because it is not clear which element from which set must be selected first. But here too, the problem can be solved in another way (using the representation of rational numbers in the form of Frey fractions). This method is not applicable for the family $\{A_r\}_{r\in\mathbb{R}}$, but it seems obvious to find the set A. The goal here is to provide a common logic language for situations like the last one.

In the early 1880s, Georg Cantor had implicitly used arguments in the proof of some theorems, which were essentially equivalent to the principle of choice, but he did not notice that he was using a new strong case principle. In 1904, Ernst Zermelo (1871-1953) after careful studies, explicitly stated the Axiom of choice and used it to prove the well-order theorem. Because no way has been found to make even the well-known set of real numbers well-ordered, despite the ruling of the well-ordered theorem, for at least six years after the appearance of this theorem, many critical articles were written about Zermelo's proof. Most of them rejected the Axiom of Choice. However, most critics had to admit that if they accepted the Axiom of Choice, they could not find fault with Zermelo's argument for the well-ordered thesis, therefore, criticizing the well-ordered thesis would lead to criticizing the Axiom of Choice. It seemed that there were only two ways:

A) Let the principle be that we accept only constructible results and do not accept purely existential results, then the methods and areas of mathematics are so limited that, outside of calculus, only very small areas can be examined.

B) To accept the constructible and purely existential results, including the principle of the subject of choice, and as a result, solve more problems and develop mathematics. To determine which method is wise to follow, the following two problematic questions must first be addressed:

1) Is the Axiom of Choice independent from the principles of the existing subject, or is it proved using other principles of the existing subject of mathematics?

2) Is the Axiom of Choice compatible with other classical principles of mathematics, or may adding the Axiom of choice to other principles of classical mathematics cause a contradiction?

Many mathematicians tried hard to find answers to these two questions. Several years later, in 1938, Kurt Gödel (1906 - 1978) answered the second question by proving that adding the Axiom of choice to other existing principles of mathematics does not create any contradiction. Gödel's discovery gave a lot of confidence to the mathematical community and especially to the users of the choice axiom. But the research to answer the first question continued. Finally, in 1963, Paul Cohen completely answered the question. He proved that the Axiom of choice is independent of other existing principles. In other words, the Axiom of Choice cannot be proved as a theorem using the principles of classical mathematics.

Today, the Axiom of Choice is accepted as a new principle, and this principle is used more in new real analysis, the theory of cardinal and transfinite ordinal numbers, modern algebra, and the field of topology.

The Axiom of Choice

In this part, we first define the product of the arbitrary family of sets, then we express its relationship with the Axiom of choice.

Definition. If I is a set and $\{A_i\}_{i\in I}$ is a family indexed by the elements of I, then the Cartesian product of this indexed family is shown as $\prod_{i\in I} A_i$, and We define as follows:

$$\prod_{i\in I} A_i = \{f \in (\cup_{i\in I} A_i)^I : \forall i \in I\,, f(i) \in A_i\}$$

If $I = \emptyset$ or one of A_i is empty, then this product is empty.

For example, suppose $I = \{1,2,3\}$, and the sets A_1, A_2, A_3 are given. We want to obtain $\prod_{i\in I} A_i$. According to the above definition, this product is equal to the set of all functions defined on the three-element set $\{1,2,3\}$, which is $f(1) \in A_1$, $f(2) \in A_2 and\ f(3) \in A_3$.

$$\prod_{i=1}^{3} A_i = \{f \in (A_1 \cup A_2 \cup A_3)^{\{1,2,3\}} : f(1) \in A_1\,, f(2) \in A_2\,, f(3) \in A_3\}$$

Therefore, each element of this set is a function whose three prices must be determined in the numbers 1, 2 and 3.

For simplicity, we denote $f(1)$ as $f_1, f(2)$ as f_2 and $f(3)$ as f_3. Basically, we display the function as $f = (f_1, f_2, f_3)$. From here, the following equality is obtained.

$$\prod_{i=1}^{3} A_i = \{(f_1\,, f_2\,, f_3) : f_1 \in A_1\,, f_2 \in A_2\,, f_3 \in A_3\} = A_1 \times A_2 \times A_3$$

From here we can conclude that the product of a family of sets is the generalization of the Cartesian product of a finite number of sets.

Therefore, any arbitrary element of the product $\prod_{i\in I} A_i$ is also shown as $(f_i)_{i\in I}$

Now we provide a detailed definition of the Axiom of Choice.

Definition (Axiom of Choice). If $I \neq \emptyset$ and for each $i \in I, A_i \neq \emptyset$, then $\prod_{i\in I} A_i \neq \emptyset$. it can be expressed as the product of the non-empty family of non-empty sets is non-empty.

In the special case, if $i \in I \neq \emptyset, A_i = A \neq \emptyset$, then $A^I \neq \emptyset$.

In other words, if A is a non-empty set consisting of non-empty sets, then there exists a set C such that each element of A contains at least one element.

Definition. The choice function for a non-empty set A is a function

$$f: P(A) \setminus \{\} \to A$$

Such that, For every $B \in P(A) \setminus \{\}$, $f(B) \in B$.

The Axiom of Choice states that if A is a non-empty set consisting of non-empty sets, then there exists a function

$$f: A \to \cup_{A_i \in A} A_i \ ,$$

Such that for each $X \in A$, $f(X) \in X$.

In this section, for a better understanding, we recall the following definitions:

Definition. If $(P, \leq)$ is a partially ordered set and $A \subseteq P$, then $u \in P$ is called an upper bound of A, whenever for each $x \in A$, we have: $x \leq u$, element m of $A(P)$ is called a maximal element $A(P)$, whenever the equality $x = m$ results from two conditions $(x \in p) x \in A$ and $m \leq x$. Similarly, the lower bound and minimum of a set are defined. In this article, the only order on the family of sets is the order $\subseteq$.

Now we will define the family of finite family.

Definition. The family $\mathfrak{J}$ of sets is called a finite characteristic family, whenever $A \in \mathfrak{J}$ if and only if all finite subsets of A are in $\mathfrak{J}$.

Let $(P, \leq)$ be a partially ordered set. A subset C of P is called a chain whenever any two elements of C are comparable. This means that for every two elements of C such as c_1 and c_2, $c_1 \leq c_2$ or $c_2 \leq c_1$ must. If $\mathfrak{R}$ is a family of sets, then it is sorted by order $\subseteq$. A subfamily C of $\mathfrak{R}$ is called a chain in $\mathfrak{R}$, whenever for both elements C_1 and C_2 of C, $C_1 \subseteq C_2$ or $C_2 \subseteq C_1$.

For example, let $\mathfrak{R}$ be the collection of all finite subsets of $\mathbb{N}$. This characteristic family is not finite, because all finite subsets of $\mathbb{N}$ are in $\mathfrak{R}$, but $\mathbb{N}$ is not in $\mathfrak{R}$ (the set of natural numbers is infinite).

Lemma. Suppose $\mathfrak{J}$ is a finite characteristic family of sets and C is a chain in it, then $\cup C \in \mathfrak{J}$.

Proof. It suffices to show that every finite subset of $\cup C$ is in $\mathfrak{J}$. Suppose $\{c_1, c_2, \dots, c_n\}$ is an arbitrary finite subset of $\cup C$. Therefore, the sets C_1, C_2, ... and C_n exist in C such that $c_1 \subseteq C_1$, $c_2 \subseteq C_2$, ... and $c_n \subseteq C_n$. But C_1, C_2, ... and C_n are comparable. So one of them like C_{i_0} includes the others. From here we can conclude that $\{c_1, c_2, \dots, c_n\} \subseteq C_{i_0}$. Since C_{i_0} is an element of the finite characteristic family of $\mathfrak{J}$, then every finite subset of it including $\{c_1, c_2, \dots, c_n\}$ is in $\mathfrak{J}$ and the theorem is proved.

The Axiom of Choice and its equivalent principles

Perhaps more than the obvious Axiom of Choice, its non-obvious equivalents are used.

In this section, we state each of these principles and prove their equivalence.

Tukey's Lemma. Every finite characteristic family has a maximal element.

Because the members of the family are characteristic of finite sets, the order on it is $\subseteq$. The maximum element in this family; That is, a set like M from that family such that if another set from that family like A were true in the condition $M \subseteq A$, then $A = M$.

For example, suppose $\mathfrak{R}$ is the family of all finite subsets of $\mathbb{N}$. We have already seen that this characteristic family is not finite. This family also does not have a maximal element, because if the finite set is its maximal element, by adding another element of natural numbers to it, we will reach a larger set, which is a contradiction.

Let $(P, \leq)$ be a partially ordered set, say C is a maximal chain in it Whenever C is a chain, the resulting set by adding another member of P is not a chain. This means that the new element is not comparable to at least one of the elements of C.

Hausdorff's Maximality Principle. Every partially ordered set has a maximal chain.

Zorn's Lemma. Every ordered set in which every chain has an upper bound has a maximal element.

Well-ordered theorem. every set can be well-ordered

In the sense that for every arbitrary set there is an arrangement with which that set is well-ordered.

In the previous sections, we stated five principles, although we may not have given them the name of the principle. For example, we called the name of one of them Tukey's Lemma, while we should have said Tukey's Principle. It has been proven that the Axiom of Choice is independent of other mathematical principles and its acceptance and rejection does not affect the science of mathematics.

Of course, accepting one of these principles and as a result accepting the other equivalent principles (according to the next theorem) makes mathematical proofs easier. At the elementary math level, we consider these principles to be known.

Theorem. The following principles are equivalent.

A: Axiom of Choice **B:** Tukey's Lemma **C:** Hausdorff Maximality Principle **D:** Zorn's Lemma **E:** Well-ordered theorem

Proof. It is enough to show that:

$$\boldsymbol{A} \Rightarrow \boldsymbol{B} \Rightarrow \boldsymbol{C} \Rightarrow \boldsymbol{D} \Rightarrow \boldsymbol{E} \Rightarrow \boldsymbol{A}$$

The longest part of the proof is the first stage of the proof, the proofs of the subsequent parts are simpler.
In this section, we first define the f-inductive subfamily for the function f from a family of sets to a subfamily of it, then we prove the first part (Tukey's Lemma is the result of the Axiom of Choice).
Definition. Suppose $\mathcal{A}$ is a family of sets and $\mathcal{B}$ is a subfamily of it and f is a function from $\mathcal{A}$ to $\mathcal{B}$. A subfamily $\oint$ of $\mathcal{A}$ is called f- induced, whenever the following three conditions apply:
a: $\emptyset \in \oint$ b: $f(\oint) \subseteq \oint$ c: If β is a chain in $\oint$, then $\cup \beta \in \oint$
part b It shows the fact that if $A \in \oint$, then $f(A) \in \oint$.
For example, suppose $\mathcal{A}$ is the family of all finite subsets of natural numbers, $f: \mathcal{A} \to \mathcal{A}$ is assumed by the rule $f(A) = A$. Family, $\mathcal{A}$ is not f- induced. Because with chains
$\beta = \{\emptyset, \{1\}, \{1,2\}, \dots, \{1,2,\dots,n\}, \dots\}$ is in $\mathcal{A}$,but $\cup \beta = \mathbb{N}$ is not in $\mathcal{A}$.
Notice: $\mathcal{B} = \{\emptyset, \{1\}\}$, is f- induced.
Now we are going to prove the first part of the theorem (Tukey's Lemma results from the Axiom of Choice).
Suppose that Tukey's lemma is not true, then there exists a non-empty and finite characteristic family called Ƒ that does not have a maximal element.
Therefore, for every $F \in$ Ƒ, the family $A_F$is nonempty as defined below

$$\mathcal{A}_F = \{E \in Ƒ : F \subsetneq E\}$$

Since $\emptyset \in$ Ƒ, then it is clear that $\{\mathcal{A}_F\}_{F\in Ƒ}$ is a nonempty family of nonempty sets. Therefore, the following non-empty choice function is available:

$$f: Ƒ \to \cup_{F\in Ƒ} \mathcal{A}_F : \; f(F) \in \mathcal{A}_F$$

According to the definition of f, for each $F \in$ Ƒ, we have: $F \subsetneq f(F)$. On the other hand, according to the finiteness characteristic of Ƒ and the definition of f, Ƒ is f-induced. Suppose $\mathcal{u}$ is the class of all families of f- induced, it is clear that Ƒ $\in \mathcal{u}$, therefore, $\oint_0$, the intersection of all families in u, is non-empty.

$$\oint_0 = \cap \mathcal{u} = \{A: \forall \mathcal{A} \in \mathcal{u}: A \in \mathcal{A}\}$$

$\oint_0$ is the smallest family f- induced, as a result, for each f- induced $\mathcal{A}$, we have: $\oint_0 \subseteq \mathcal{A}$, we define the collection of $\mathcal{H}$ as follows:

$$\mathcal{H} = \{A \in \oint_0 : B \in \oint_0, B \subsetneq A \Rightarrow f(B) \subseteq A\}$$

We prove that if $A \in \mathcal{H}$ and $C \in \oint_0$, then $C \subseteq A$ or $f(A) \subseteq C$. To prove, for each A member of $\mathcal{H}$, we define the family y_A as follows:

$$y_A = \{C \in \oint_0 : C \subseteq A \quad or \quad f(A) \subseteq C\}$$

y_A is an f- induced family of subset $\oint_0$, therefore, $y_A = \oint_0$. Now we show that $\mathcal{H} \subseteq \oint_0$, is f- induced and as a result $\mathcal{H} = \oint_0$.

A: $\emptyset$ does not have any special subset and according to the definition of $\mathcal{H}$ and the law of antecedent termination, $\emptyset \in \mathcal{H}$.

B: Suppose $A \in \mathcal{H}$ we show that f$(A) \in \mathcal{H}$. For this, we must show that if $B \in \oint_0$, and $B \subsetneqq f(A)$, then $f(B) \subseteq f(A)$. Suppose $B \in \oint_0$ and $B \subsetneqq f(A)$. Since $B \in \oint_0 = y_A$, thus $B \subseteq A \ or\ f(A) \subseteq B$.

But according to $B \subsetneqq f(A)$, the state $f(A) \subseteq B$ is impossible, and as a result, $B \subseteq A$. From here, we have two situations, $B \subsetneqq A \ or\ B = A$. If $B \subsetneqq A$ then according to the definition of $\mathcal{H}$, $f(B) \subseteq A \subseteq f(A)$ and therefore $f(B) \subseteq f(A)$ and therefore, $f(A) \in \mathcal{H}$, if $B = A$, clear is that $f(B) \subseteq f(A)$ and therefore, $f(A) \in \mathcal{H}$. So if $A \in \mathcal{H}$, then $(A) \in \mathcal{H}$.

C: It should be shown that if β is a chain in $\mathcal{H}$, then $\cup\beta \in \mathcal{H}$ and assume that $B \in \oint_0$ and $B \subsetneqq \cup\beta$. Because $B \in \oint = y_A$ for every, $A \in \beta$, then two situations happen:

$$1)\ B \subseteq A;\ \exists A \in \beta \quad or \quad 2)\ f(A) \subseteq B;\ \forall A \in \beta$$

If the second situation happens, then we reach the following contradiction:

$$\mathrm{B} \subsetneqq \cup\beta \subseteq \cup_{\mathrm{A}\in\beta} \mathrm{f(A)} \subseteq \mathrm{B}$$

Therefore, the first state occurs. So for an $A \in \beta$, we have: $B \subseteq A$. If $B \subsetneqq A$, Since $A \in \mathcal{H}$, then $f(B) \subseteq A \subseteq \cup\beta$ and therefore $f(B) \subseteq \cup\beta$, from here the relation $\cup\beta \in \mathcal{H}$ results.

If $B = A$, then $B \in \mathcal{H}$ and $\cup\beta \in \oint_0 = y_B$. This is impossible, because $f(B) \subseteq \cup\beta$. From here it is concluded that $\cup\beta \in \mathcal{H}$ and $\mathcal{H}$ is an f- induced. Therefore, $\mathcal{H} = \oint$.

Now we complete the proof. If $A \in \oint_0 = \mathcal{H}$ and $B \in \oint_0 = y_A$, then $B \subseteq A$ or $f(A) \subseteq B$. But with the help of the second relation and $A \subseteq f(A)$, the relation $A \subseteq B$ is obtained. So $\oint_0$ is a chain. If $M = \cup \oint_0$, Since $\oint_0$ is of the f- induced family, then $M \in \oint_0$. But $M \subsetneqq f(M) \in \oint_0$. Since M is the union of all elements of $\oint_0$ and since $f(M) \in \oint_0$, therefore $f(M) \subseteq M$ and this is a contradiction .

In this part, we prove the following three conclusions:

Tukey's lemma ⇒ Hausdorff's maximality principle ⇒ Zorn's lemma
⇒ Well-ordered theorem ⇒ Axiom of Choice

Proof (Tukey's Lemma gives the Maximality principle).

Let $(P, \leq)$ be a non-empty partially ordered set. If we consider y as the family of all chains in P, then it is clear that y is a finite characteristic family. From here it

Proof (Maximality principle gives Zorn's Lemma).

Let $(P, \leq)$ be a non-empty partially ordered set, each chain of which has an upper bound. According to Hausdorff's maximality principle, there is a maximal chain C in P. Suppose m is an upper bound of C, we show that m is a maximal element of P. Suppose it is not so, then member x exists in P such that $x > m$. It is clear that $C' = C \cup \{x\} \neq C$ is a chain in P that contradicts the maximality of C.

Proof (The Well-ordered theorem results in the Axiom of Choice).

Suppose $\{A_i\}_{i \in I}$ is a non-empty family of non-empty sets. We well-order the set $D = \cup_{i \in I} A_i$. Now the function $f: I \to D = \cup_{i \in I} A_i$ with the rule $f(i) = minA_i$ is in $\prod_{i \in I} A_i$.

To prove (the well-order principle is the result of Zorn's Lemma):

By placing an order on non-empty partially ordered family sets, we show that Zorn's Lemma results in the well-ordering theorem. Here, the ordered sets are shown as pairs consisting of the set and the order on it. For example, a good order on a one-element set $P = \{x\}$ must contain $x \leq x$, and as a result, our order will be $\leq = \{(x, x)\}$. So the latter ordered set can be represented as $(\{x\}, \{(x, x)\})$.

Suppose S is a set. We take $\mathcal{Z}$ to be the family of all well-ordered subsets of S such as W with order $\leq W$

According to the previous explanation, $(\{x\}, \{(x, x)\}) \in \mathcal{Z}$ for each $x \in S$, now we place the order $\preccurlyeq$ on $\mathcal{Z}$. Suppose that $(W_1, \leq_1)$ and $(W_2, \leq_2)$are in , we define $(W_1, \leq_1) \preccurlyeq (W_2, \leq_2)$ which means that $W_1 = W_2$ and $\leq_1 = \leq_2$ or that element a exists in S that:

$$W_1 = \{x \in W_2 : x \leq a \ , x \neq a\}$$

And, $W_1 \subseteq W_2$, in this case we say $(W_2, \leq_2)$is a continuation of $(W_1, \leq_1)$

Proof (The Well-ordered theorem is a result of Zorn's Lemma)

Suppose S is an arbitrary non-empty set, suppose $C = \{(W_i, \leq_i)\}_{i \in I}$ is a chain in $\mathcal{Z}$ according to the order $\preccurlyeq$. We put $= \cup_{i \in I} W_i$, $\leq \ = \ \cup_{i \in I} \leq_i$. The set $(W, \leq)$ is well ordered. Therefore, according to Zorn's Lemma, $\mathcal{Z}$ has a maximal element called $(W_\circ, \leq_\circ)$. $W_\circ = S$ because otherwise, for $x \in S - W_\circ$, the well-ordered $\leq_\circ$ exists on $W_\circ \cup \{x\}$ as follows

$$\leq_x = \leq_\circ \cup \{(w, x) : w \in W_\circ\}$$

This content is contradictory to the maximality of $(w_\circ, \leq_\circ)$.
And this proof completes the case.

Applications

In this section, with the help of the Axiom of Choice and its equivalents, we prove some important theorems of other branches of mathematics.

Theorem. If $f: A \to B$ is surjective, it has a right inverse.

Proof. Since f is surjective, we consider the non-empty collection of non-empty sets $\{f^{-1}(\{b\})\}_{b\in B}$. Therefore, we consider the choice function

$$g: B \to \cup_{b\in B} f^{-1}(\{b\}$$

With the condition $g(b) \in f^{-1}(\{b\})$. But the definition of, each $a \in f^{-1}(\{b\})$ is true under the condition, $f(a) = b$. Among other things, for $a = g(b)$, the relation $f(g(b)) = b$ is obtained. From here it is clear that $g : B \to A$ is the right inverse of f.

Theorem. If A is a nonempty set, then the function $f: A \to B$ is surjective if and only if $= B$.

Proof. If f is surjective, then there exists a function $g: B \to A$ such that $fog = \mathrm{id}_B$, then, for every $y \in B$, $f\big(g(y)\big) = y$. we put $x = g(y)$, therefore, $f(x\ ) = y$ and $B \subseteq Imf$, the relationship $Imf \subseteq B$ is also clear, so $B = Imf$.

On the contrary, assume that $B = Imf$.

For every $y \in B$, the set

$$T_y = \{x \in A: f(x) = y\}$$

is non-empty.

According to the Axiom of Choice in the case of set A, this set will have a choice function $\varphi: P(A) \setminus \{\ \ \} \to A$. Now we define the function $g: B \to A$ as follows

$$\forall\, y \in B \ \ (g(y) = \varphi(T_y))$$

To complete the proof, it is enough to check the relation that $fog = \mathrm{id}_B$

For each $y \in B$ we have:

$$(fog)(y) = f\big(g(y)\big) = f\left(\varphi(T_y)\right)$$

On the other hand, considering the definition of the choice function, if $\varphi\big(T_y\big) = x$ then $x \in T_y$. So

$$(fog)(y) = f(x) = y$$

As another application, we show that every vector space V on the scalar field $\mathbb{F}$ has a basis.

We know that for the vector space V on the field $\mathbb{F}$, $B \subsetneqq V$ is called a basis for V, whenever the elements of B are linearly independent, and for each $\vec{v} \in V$, finite -m-element subset (m depends on the vector v) of B say $B_v = \overrightarrow{\{v_1, \vec{v}_2, \dots, \vec{v}_m\}}$ and m scalars f_1, f_2,... and f_m from $\mathbb{F}$ exist such that:

$$\vec{v} = f_1\vec{v}_1 + f_2\vec{v}_2 + \cdots + f_m\vec{v}_m$$

Theorem. Every vector space V on the scalar field $\mathbb{F}$ has a basis.

Proof. We consider the collection A consisting of all independent subsets of V, the family A is characteristically finite, because if the elements of the set are linearly independent, the elements of each of its subsets, including its finite subsets, are linearly independent and vice versa. According to the definition, if every finite subset of a set is linearly independent, then the elements of the set itself are also linearly independent. According to Tukey's lemma, A has a maximal element B. According to the definition of A, the elements of B are independent. Now we have to show that every element of V like $\vec{v}$ is a linear combination of finite elements of B. If $\vec{v} \in B$, then the sentence is obvious, because $\vec{v} = 1\,\vec{v}$. Suppose $\vec{v} \notin B$. Considering that B is a maximal linearly independent set, the elements of the set $\mathcal{H} = B \cup \{\vec{v}\}$ are dependent. Therefore, there are finitely many elements of $\mathcal{H}$ that are dependent, one of them must be $\vec{v}$. So this dependent set is $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \dots, \vec{v}_m\}$, which $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \dots, \vec{v}_m\}$ is a subset of B. From here the scalars $f_\circ \neq 0$, f_1, f_2, … and f_m exist such that:

$$f_\circ\vec{v} + f_1\vec{v}_1 + f_2\vec{v}_2 + \cdots + f_m\vec{v}_m = 0$$

Given that $f_\circ \neq 0$, it is clear that $\vec{v}$ is a linear combination of elements of B, and therefore B is a basis for V.

Theorem. Suppose V is a vector space on the field $\mathbb{F}$ and S is a linearly independent subset of V. Then there is a basis for V containing the set S.

Proof. We take the set A consisting of all linearly independent sets in V that contain S. It is clear that $S \in A \neq \emptyset$. We show that every chain in A has an upper bound. Let $\{H_i\}_{i\in I}$ be a chain in A, we show that $H = \cup_{i\in I} H_i$ is a linearly independent set. Suppose $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\} \subseteq H$, from here n index i_1, i_2, ... and i_n of I exists such that $\vec{v}_t \in H_{i_t}$, because H_{i_1}, H_{i_2} , …. and H_{i_n} are elements of the chain $\{H_i\}_{i\in I}$. So compared, one of them is bigger than the others like H_{i_k}. From here, the following relationship is obtained:

$$\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\} \subseteq H_{i_k}$$

And since the elements of H_{i_k} are independent, then the elements $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m\}$ are independent, so H is a linear independent set.

According to Zorn's Lemma, A has a maximal element. This maximal element is a basis for V

CONCLUSION

From the topics examined here, it is clear that the Axiom of Choice is one of the basic and important principles in mathematics, on which many other mathematical results are expressed and proved.

It has been proven that the Axiom of Choice is independent of other mathematical principles and its acceptance and rejection do not affect the science of mathematics. This principle is equivalent to several other famous principles and accepting one of these principles and as a result, accepting the rest of its equivalent principles makes mathematical proofs easier.

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