REPRODUCING KERNEL HILBERT SPACE METHOD FOR SOLVING ABEL'S INTEGRAL EQUATIONS

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ABSTRACT

The Integration Equations are one of the significant & essential phenomena and abstractions for the sake of numerous types of problem-solving in mathematics and have had various applications in areas of different fields. In fact, have a high theoretical & applicability importance. Even so, in this Article, a detailed research is carried out into the "Abel Integral Equation" utilizing the Reproducing Kernel Hilbert Space Method (RKHS) which is one of the strong, concise & perfect methods particularly for simplification & solution of the Abel Integral Equation. The main purpose of this research is to seek the numerical solution for the Abel Integral Equation by RKHS Method.

Keywords: Abel Integral Equation, Generalized Abel Integral Equation, Reproducing Kernel Hilbert Space Method

INTRODUCTION

The Integration Equations have an ample applicability in all areas of the daily life and have had numerous plentiful applications in various fields of science such as the seismology, radio-astronomy, electron emission, plasma diagnostics, X-Ray radiography, fiber evaluation, scattering of the diseases infection, semiconductor design, heat transfer & crystal growth. If the unknown function of $u(x)$ in a differential equation comes under the symbol of Integral, then this type of Equation is called "The Integral Equation" [3-7]. Note that the Reproducing Kernel Hilbert Space Method (RKHS) has been applied for the sake of Solution of a specified class of integration equations for electro-magnetic non-linear problems as well [1]. In order to solve the "generalized Abel integral equation", it is needed to make a blend of the Laplace transform and Jacobi collocation Methods and in the meantime the estimation of its error can also be calculated [12]. The solution of Abel integral equation together with the weakly singular kernels has been considered based on the remaining functions of (RBF) and the results of the method have been compared with the methods of HPM & ADM [13]. It is to be noted that the second type of the Abel integral equation has been considered based on the Jacobi polynomials and Jacobi spectral collocation method [14]. Approximate method solution – analysis of integral equation by Taylor series [15]. Solution of generalized Abel integral equation by using of the HPM & MHPM methods [16]. The numerical solution of Abel integral equation has been proved upon the approximate amounts of the Legendre Wavelets [17] and by the method of Laplace transform [4], utilizing of the Taylor series, utilizing of the approximate method of “Pade” and its convergence [2]. The analytical solution of the Abel integral equation has been assessed in astronomical physics by the method of Laplace transforms and its numerical solution has been assessed by using of the Homotopy method [3]. The combined reproducing Kernel Method and Taylor series to solve the non-linear integral equation together with weakly kernels [8], stability Analysis, Error Analysis of the reproducing kernel Hilbert space method for the sake of solution of second type of singular Voltaire integral equation on graded mesh [9], First type of singular integral equation, Cauchy type integral equation by RKHS method and the estimation of its error have also been assessed [10]. The numerical solution of type-II Voltaire Integral Equation with the weakly singular kernel has been worked-out based on the Block-Pulse functions & error analysis [18]. The Fredholm integral equation and Volterra-Fredholm integral equations; These equations study & examine the errors and stability by using of the Reproducing Kernel Hilbert Space method (RKHS) and one of the goodness of this method is the rapidity of its convergence [11].

In 1823 A.D Abel studied the motion of a particle, sliding downward under the exertion of gravity load along an unknown smooth curve in a right-angled geometrical Plane. Assume that the particle is located motionless at point P along the unknown smooth curve having the height of x from the 0rigin and starts sliding downward under the exertion of gravity load until the origin ($Point\, O$) whose height is presumed to be zero. Now we calculate the motion equations of the particle under exertion of gravity load along the noted smooth curve at every assumed point in time T until the origin ($Point\, O$). Note that we assume the $x\, Axis$ is termed to be the height and the $y\, Axis$ is assumed to be the horizontal distance. Pretend the points $P(x,y)$ $Q(\xi,\eta)$ to be located on the smooth curve and s denotes the OQ. We probably find the kinetic and potential energy at any moment for designated point of Q. Even so, the summation of both energies become equal to a constant and it is signified as the following physical expression:

$$K.E + P.E = C$$
$$1/2\,mv^2 + mg\xi = C$$
$$\text{or } 1/2\,v^2 + g\xi = C$$

Hence, m is the mass of the particle, $v(t)$ is the velocity of particle in time t at Q, g is termed to be the gravity and ξ is the height of point Q. First, at $v(0) = OP$ the height of particle is x, even so, the constant C is equal to gx as indicated at the below:

$$1/2\,v^2 + g\xi = gx$$
$$v = \pm\sqrt{2g(x-\xi)}$$

Since; $\frac{ds}{dt} = v$ is the velocity along the distant s of the smooth curve, even so,

$$\frac{ds}{dt} = \pm\sqrt{2g(x-\xi)}$$

Consider the negative value of $\frac{ds}{dt}$ and after performance of mathematical calculations and integration of both sides of the equation from point P to point Q, the following results are gained:

$$\int_P^Q dt = -\int_P^Q \frac{ds}{\sqrt{2g(x-\xi)}} \Rightarrow t = -\int_P^Q \frac{ds}{\sqrt{2g(x-\xi)}}$$
$$\int_P^O dt = -\int_P^O \frac{ds}{\sqrt{2g(x-\xi)}}$$
$$T = -\int_O^P \frac{ds}{\sqrt{2g(x-\xi)}}$$

If the curve is already specified, then s can express the ξ. Therefore, ds can also express the $d\xi$.

Pretend:

$$ds = u(\xi)d(\xi)$$

Hence, the latest equation forms as following:

$$T = \int_0^x \frac{u\xi d\xi}{\sqrt{2g(x-\xi)}}$$

The equation of the curve in which T is the descending time is termed to be the function of x and is denoted in $f(x)$. Even so,

$u(x)$ is an unknown function and the following equation is achieved:

$$f(x)=\int_0^x \frac{u\xi d\xi}{\sqrt{2g(x-\xi)}}=\int_0^x K(x,\xi)u(\xi)d(\xi)$$

This is a Type-I Linear integral equation. In here, the kernel of the integral equation is at the below:

$$K(x,\xi)=\frac{1}{\sqrt{2g(x-\xi)}}.$$

It is to be noted that the Abel integral equation is common under the name of type-I Voltaire integral equation as well. Note that Abel introduced a more overall singular integral equation which named it “generalized Abel integral equation” [5-7].

$$f(x)=\int_0^x \frac{1}{(x-t)^{\alpha}}u(t)dt, \quad 0<\alpha<1.$$

THE STRUCTURE OF METHOD

To solve the problem, we use the following Reproducing kernel.

$$R_y(x)=\begin{cases}1-a+x, & x\le y,\\ 1-a+y, & x>y,\end{cases} \tag{1}$$

Therefore for additional information consult [4-6]. Now we analyze Abel integral equation in the below mentioned form,

$$u(x)=f(x)+Gu(x) \tag{2}$$

Then

$$Gu(x)=\int_0^x \frac{u(t)}{\sqrt{x-t}}dt, \tag{3}$$

Thus, the solution (2) in the Reproducing kernel space $W_2^1[a,b]$ with assuming the $\psi_i(x)=R_{x_i}(x)$, therefore $\{x_i\}_{i=1}^{\infty}$ in interval $[a,b]$ is dense. We can write it in this form:

$$\langle u(x),\psi_i(x)\rangle_{W_2^1}=u(x_i) \tag{4}$$

Theorem 2.1 Let $\{x_i\}_{i=1}^{\infty}$ be dense in the interval $[a,b]$. If the equation (2) has a unique solution, then the solution satisfies the form

$$u(x)=\sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\left(f(x_k)+Gu(x_k)\right)\overline{\psi}_i(x) \qquad (5)$$

Proof. Assume $u(x)$ be the solution of Eq. (2). $u(x)$ is expanded in Fourier series, it has

$$u(x)=\sum_{i=1}^{\infty}\left\langle u(x),\overline{\psi}_i(x)\right\rangle_{W_2^1}\overline{\psi}_i(x)=\sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\left\langle u(x),\psi_k(x)\right\rangle_{W_2^1}\overline{\psi}_i(x)$$

$$=\sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\left\langle f(x)+Gu(x),\psi_k(x)\right\rangle_{w_2^1}\overline{\psi}_i(x)$$

$$=\sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\left(f(x_k)+Gu(x_k)\right)\overline{\psi}_i(x) \qquad \square$$

The proof is complete.

IMPLEMENTATIONS OF THE METHOD

Here, a method of solving (5) of (2) is given in the reproducing kernel space [11]. Rewrite (5) as

$$u(x)=\sum_{i=1}^{\infty}A_i\,\overline{\psi}_i(x)$$

Where

$$A_i=\sum_{k=1}^{i}\beta_{ik}\left(f(x_k)+Gu(x_k)\right).$$

In fact, A_i is unknown. A_i Is approximated by known B_i . For a numerical computation, let initial function $u_1(x)=f(x)$ and the n-term approximation to $u(x)$ is defined by

$$u_{n+1}(x)=\sum_{i=1}^{n}B_i\,\overline{\psi}_i(x), \qquad (6)$$

Where

$$
\begin{aligned}
B_1 &= \beta_{11}\left(Gu_1(x_1)+f(x_1)\right),\\
u_2(x) &= B_1\overline{\psi}_1(x),\\
B_2 &= \sum_{k=1}^{2}\beta_{2k}\left(Gu_2(x_k)+f(x_k)\right),\\
u_3(x) &= B_1\overline{\psi}_1(x)+B_2\overline{\psi}_2(x),\\
&\dots \qquad\qquad (7)\\
B_n &= \sum_{k=1}^{n}\beta_{nk}\left(Gu_n(x_k)+f(x_k)\right),\\
u_{n+1}(x) &= \sum_{k=1}^{n}B_k\,\overline{\psi}_k(x).
\end{aligned}
$$

APPLICATIONS AND NUMERICAL RESULTS

For the solution of weakly singular Abel integral equation of the second kind. Linear, nonlinear equations and the nonlinear generalized Abel integral equation is analyzed by the method of reproducing kernel. That we find approximation solution, absolute error and relative error of Abel integral equation. Here the equation (2) is taken in the form of an operator and the examples are calculated by Mathematica 11.

Example 4.1: Consider nonlinear Abel integral Equation [7]. With the exact solution $u(x)=x$.

$$u(x)=x-\frac{16}{15}x^{\frac{5}{2}}+\int_0^x\frac{u^2(t)}{\sqrt{x-t}}dt \qquad 0\le x\le 1.$$

We choose $N=20$ points and find the approximation solution By (6).

Table 1. Numerical results for example 4.1

Node	True solution $u(x)$	Approximate solution	Absolute error	Relative error
0.05	0.0526316	0.0526316	6.47085×10^{-12}	1.22946×10^{-10}
0.11	0.105263	0.105263	4.23296×10^{-11}	4.02132×10^{-10}
0.16	0.157895	0.157895	1.34516×10^{-10}	8.51932×10^{-10}
0.21	0.210526	0.210526	3.16006×10^{-10}	1.50103×10^{-9}
0.26	0.263158	0.2631578	6.30124×10^{-10}	2.39447×10^{-9}

0.32	0.315789	0.315789	1.13683×10^{-9}	3.59995×10^{-9}
0.37	0.368421	0.368421	1.923×10^{-9}	5.21957×10^{-9}
0.42	0.421053	0.421053	3.12889×10^{-9}	7.43111×10^{-9}
0.47	0.473684	0.473684	5.19044×10^{-9}	1.09576×10^{-8}
0.53	0.526316	0.526316	1.22316×10^{-8}	2.324×10^{-8}
0.58	0.578947	0.578947	5.86176×10^{-8}	1.01249×10^{-7}
0.63	0.631579	0.631578	6.58986×10^{-7}	1.04339×10^{-6}
0.68	0.684211	0.684205	5.74339×10^{-6}	8.39419×10^{-6}
0.74	0.736842	0.736799	4.30481×10^{-5}	5.84224×10^{-5}
0.79	0.789474	0.789199	2.75056×10^{-4}	3.48405×10^{-4}
0.84	0.842105	0.840604	1.50164×10^{-3}	1.78320×10^{-3}
0.89	0.894737	0.887799	6.93789×10^{-3}	7.75411×10^{-3}
0.95	0.947368	0.920768	2.66006×10^{-2}	2.80784×10^{-2}

Example 4.2: Consider nonlinear generalized Abel integral equation [19].

$$u(x)=x^2-\frac{3}{16}\pi x^4+\int_0^x \frac{u^2(t)}{\sqrt{x^2-t^2}}dt \qquad 0\le x\le 1.$$

With the exact solution $u(x)=x^2$. We choose $N=20$ points for approximation solution.

Table 2. Numerical results for example 4.2

Node	True solution $u(x)$	Approximate solution	Absolute error	Relative error
0.05	0.002770	0.002773	1.51325×10^{-6}	5.46282×10^{-4}
0.11	0.011080	0.011087	6.26058×10^{-6}	5.65018×10^{-4}
0.16	0.024931	0.024946	1.49034×10^{-5}	5.97791×10^{-4}
0.21	0.044321	0.044349	2.78974×10^{-5}	6.29436×10^{-4}
0.26	0.069252	0.069298	4.58696×10^{-5}	6.62357×10^{-4}
0.32	0.099723	0.099794	6.96952×10^{-5}	6.98888×10^{-4}
0.37	0.135734	0.135835	1.00590×10^{-4}	7.41085×10^{-4}

0.42	0.177285	0.177426	1.40245×10^{-4}	7.91070×10^{-4}
0.47	0.177285	0.224568	1.91021×10^{-4}	8.51341×10^{-4}
0.53	0.277008	0.277265	2.56257×10^{-4}	9.25087×10^{-4}
0.58	0.335180	0.335521	3.40749×10^{-4}	1.01661×10^{-3}
0.63	0.398892	0.399343	4.51501×10^{-4}	1.13189×10^{-3}
0.68	0.468144	0.468743	5.99020×10^{-4}	1.27956×10^{-3}
0.74	0.542936	0.543736	7.99397×10^{-4}	1.47236×10^{-3}
0.79	0.623269	0.624347	1.07810×10^{-3}	1.72976×10^{-3}
0.84	0.709141	0.710618	1.47664×10^{-3}	2.08230×10^{-3}
0.89	0.800554	0.802619	2.06518×10^{-3}	2.57969×10^{-3}
0.95	0.897507	0.900473	2.96604×10^{-3}	3.30476×10^{-3}

Example 4.3: Consider Abel integral equation of the second kind [20].

$$u(x)=x+\frac{4}{3}x^{3/2}-\int_0^x\frac{u(t)}{\sqrt{x-t}}dt,\qquad 0\le x\le 1.$$

Where the best value of the maximum absolute error obtained in [3] was 5×10^{-6} at $N=25$and also considered in [20] by the Babenko's approach and fractional integrals expressed as,

$$\int_0^x\frac{u(t)}{\sqrt{x-t}}dt=\frac{\sqrt{\pi}}{\Gamma(1/2)}\int_0^x\frac{u(t)}{\sqrt{x-t}}dt$$

Subsequently, obtained the following series

$$\sum_{m=0}^{\infty}\frac{(\pi x)^{m/2}}{\Gamma(m/2+3/2+1)}=\mathrm{E}_{1/2,3/2+1}\left(\sqrt{\pi x}\right),$$

And the series convergence to the exact solution $u(x)=x$. Now we find the approximation solution and errors by the method of (RKHS). Here we choose $N=21$ points for solving Abel integral equation.

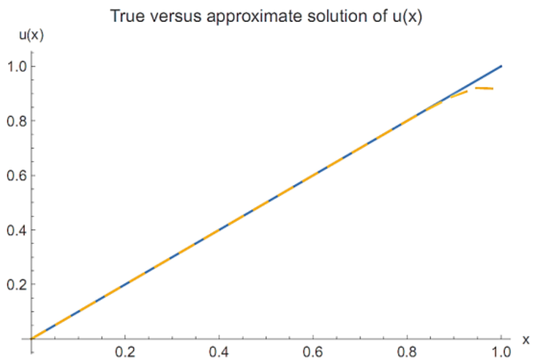
Table 3. Numerical results for example 4.3

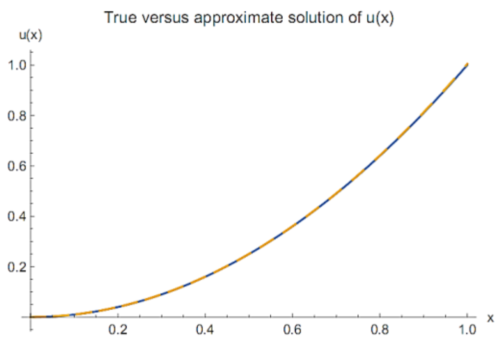
Node	True solution $u(x)$	Approximate solution	Absolute error	Relative error

0.05	0.05	0.05	1.80528×10^{-11}	3.61055×10^{-10}
0.1	0.1	0.1	1.41603×10^{-10}	1.41603×10^{-9}
0.15	0.15	0.15	5.15052×10^{-10}	3.43367×10^{-9}
0.2	0.2	0.2	3.23208×10^{-10}	1.61604×10^{-9}
0.25	0.25	0.25	3.11418×10^{-9}	1.24567×10^{-8}
0.3	0.3	0.3	1.44857×10^{-8}	4.82856×10^{-8}
0.35	0.35	0.35	5.16753×10^{-8}	1.47644×10^{-7}
0.4	0.4	0.4	1.58849×10^{-7}	3.97125×10^{-7}
0.45	0.45	0.45	4.89252×10^{-7}	1.08724×10^{-6}
0.5	0.5	0.500001	1.3479×10^{-6}	2.69581×10^{-6}
0.55	0.55	0.550003	3.41419×10^{-6}	6.20762×10^{-6}
0.6	0.6	0.600008	8.20784×10^{-6}	1.36797×10^{-5}
0.65	0.65	0.650018	1.8493×10^{-5}	2.84507×10^{-5}
0.7	0.7	0.70004	3.97055×10^{-5}	5.67222×10^{-5}
0.75	0.75	0.750081	8.14655×10^{-5}	1.0862×10^{-4}
0.8	0.8	0.800161	1.60638×10^{-4}	2.00797×10^{-4}
0.85	0.85	0.850306	3.05732×10^{-4}	3.59685×10^{-4}
0.9	0.9	0.900563	5.63211×10^{-4}	6.2579×10^{-4}

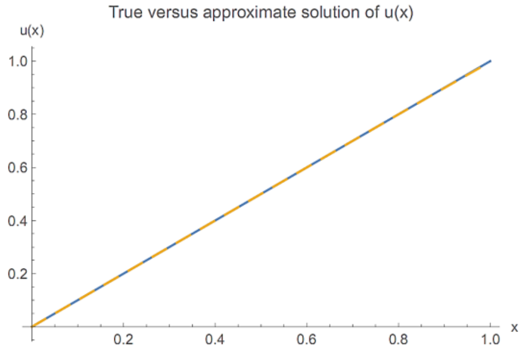
Example 4.4: Let us consider the following weakly singular Abel integral equation, With the exact solution $u(x)=x^7$ [18].

$$u(x)=x^7\left(1-\frac{4096}{6435}\sqrt{x}\right)+\int_0^x \frac{u(t)}{\sqrt{x-t}}dt$$

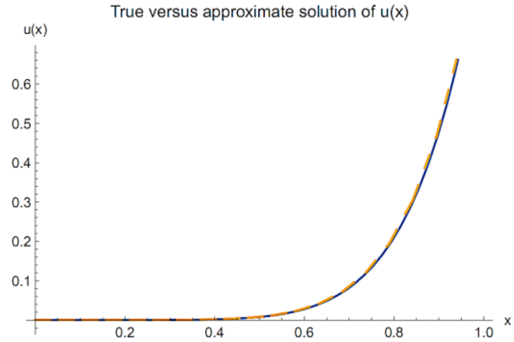




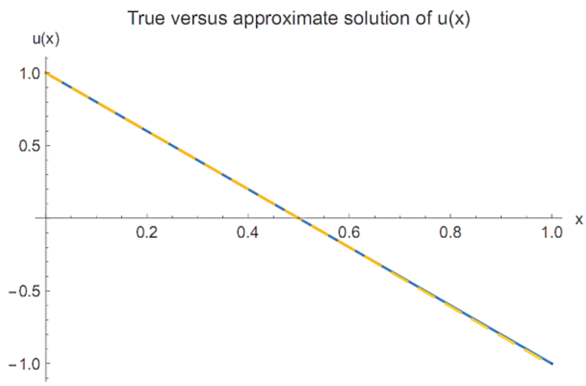
(a) Example 4.1. (b) Example 4.2.

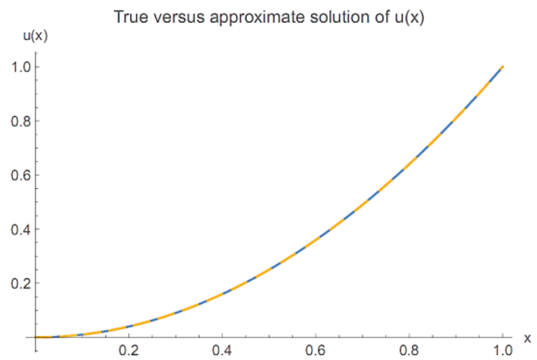


(c) Example 4.3.



(d) Example 4.4.





(e) Example 4.5. (f) Example 4.7.

Figure 1. Exact solution versus numerical solution: continuous blue line for true solution and yellow dashed line for approximation solution.

Example 4.5: Consider the following generalized weakly singular Abel integral equation of second kind with the exact solution $u(x)=1-2x$ [7].

$$u(x)=1-2x-\frac{32}{21}x^{7/4}+\frac{4}{3}x^{3/4}-\int_0^x \frac{u(t)}{\sqrt[4]{x-t}}dt$$

Table 4. Absolute error for examples 4-7 with choose $N = 20$ points calculated

Node	Example 4	Example 5	Example 6	Example 7
0.05	2.57653×10^{-10}	6.56142×10^{-14}	1.29771×10^{-4}	7.77107×10^{-5}
0.11	2.10035×10^{-8}	1.01696×10^{-13}	1.71385×10^{-4}	1.18009×10^{-4}
0.16	2.46953×10^{-7}	1.19793×10^{-13}	1.98369×10^{-4}	1.47419×10^{-4}
0.21	1.38795×10^{-6}	1.22791×10^{-13}	2.1804×10^{-4}	1.70716×10^{-4}
0.26	5.27139×10^{-6}	1.14131×10^{-13}	2.33389×10^{-4}	1.89974×10^{-4}
0.32	1.56901×10^{-5}	9.24816×10^{-14}	2.45886×10^{-4}	2.06329×10^{-4}
0.37	3.95425×10^{-5}	6.2117×10^{-14}	2.56371×10^{-4}	2.20482×10^{-4}
0.42	8.83116×10^{-5}	2.21489×10^{-14}	2.65381×10^{-4}	2.32904×10^{-4}
0.47	1.79940×10^{-4}	2.84217×10^{-14}	2.73316×10^{-4}	2.4393×10^{-4}
0.53	3.41170×10^{-4}	9.59094×10^{-14}	2.8056×10^{-4}	2.53802×10^{-4}
0.58	6.10427×10^{-4}	4.60743×10^{-15}	2.87622×10^{-4}	2.6271×10^{-4}
0.63	1.04134×10^{-3}	4.10116×10^{-13}	2.95336×10^{-4}	2.70797×10^{-4}

Example 4.6: Consider the following weakly singular Abel integral equation of second kind with the exact solution $u(x) = x^2$ [12], [17].

$$u(x) = x^2 + \frac{16}{15}x^{5/2} - \int_0^x \frac{u(t)}{\sqrt{x-t}}dt$$

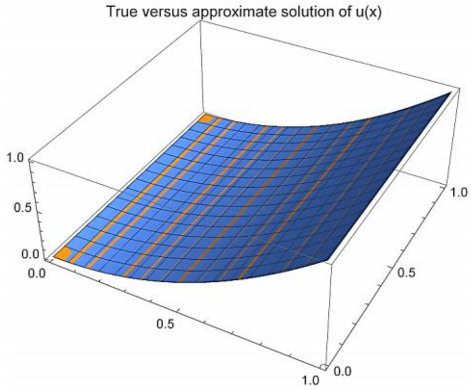


Fig. 6: The exact and numerical solutions curves for example 4.6

Example 4.7: Consider the following generalized Abel integral equation of second kind with the exact solution $u(x) = x^2$ [16].

$$u(x) = x^2 + \frac{27}{40}x^{8/3} - \int_0^x \frac{u(t)}{\sqrt[3]{x-t}}dt$$

CONCLUSION

In this Article all of the specified mathematical and calculus features such as the Abel integral equation, weakly-singular integral equation, generalized Abel integral equation, introduction of reproducing kernel Hilbert space method for the sake of numerical solution of Abel integral equation and the approximate value, with respect to the exact solution, absolute error and relative error, for the Abel integral equation utilizing the reproducing kernel Hilbert space (RKHS) method; have been studied and considered.

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