AN ANALYTICAL DESCRIPTION OF THE MANDELBROT SET FOR THE SOME TWO DIMENSIONAL MAPPINGS

Shavkat Jumabayevich Seytov
Tashkent State University of Economics
sh-seytov@mail.ru

Muxlisaxon Olimjon qizi Oxunova
Fergana University
muxlisaxonoxunova@yahoo.com

ABSTRACT

In this paper proposes one of generalizations ofthe following the mappings $x \to x^3 + c$ to the multi-dimensional case. An analytical description of the Mandelbrot set for the twodimensional case is obtained. The properties of Julia sets are studied.

INTRODUCTION

The study of dynamics of the mapping $z \to z^3 + c$ on complex plane to itself and its various generalizations aredevoted hundreds papers beginning the classical researches ofMandelbrot, Devaney, Andrien Douady[3,4,9].

In the present time the theory of one dimensional mapping is the most learned part of the general theory of dynamical systems.

This paper is devoted to one of the possible generalizations case of the problem of learning properties of the mappings $x \to x^3 + c$.

Let $x = (x_1, x_2, ..., x_n) \in R^n$, $c = (c_1, c_2, ..., c_n) \in R^n$, $I = \{1,2,..., n\}$ and $\pi : I \to I$ some permutations. We callthis mapping

$$x_k = x^3_{\pi(k)} + c_{\pi(k)},\ k = \overline{1,n} \tag{1}$$

on R^n to itself **is multi – dimensional case** of problem of the mappings $x \to x^3 + c$. If the permutation π expansions in product of the several cycles, R^n also expansions Cartesian product **of sub spaces every from** invariant at (1) mapping. Therefore dynamical properties are also defined by Cartesian product of dynamical properties of invariant **sub spaces.** Hence it is enough to learn when π - cyclical permutation has maximal length.

First we learn when $n = 2$. In this case mapping (1) is

$$\begin{cases} x' = y^3 + c_1 \\ y' = x^3 + c_2 \end{cases} \tag{2}$$

where $(x,y)\in R^2$ and $(c_1,c_2)\in R^2$.

1. Fixed points of the mapping (2).

For finding fixed points of the mapping (2) necessary to solve thefollowing equation

$$x=\left(x^3+c_2\right)^3+c_1=x^9+3x^6c_2+3x^3c_2^2+c_2^3+c_1. \qquad (3)$$

Let $f(x)=x^9+3x^6c_2+3x^3c_2^2-x+c_2^3+c_1$ the polynomial of ninth degree and has two parameters c_1 and c_2. We know for some c_1 and c_2 equation (3) has at least one real root. Let denote it by $x_1=8t\in R$.

Lemma 1. Polynomial $f(x)$ has no four multiple complex roots.

Proof. Let z_0 is the four multiple complex root of $f(x)$. Then Let $\overline{z_0}$ is also the multiple complex root of $f(x)$. By theorem Viet $x_1+\sum_{i=1}^{8}z_i=x_1+4z_0+4\overline{z_0}=0$, hence $\operatorname{Re} z_0=-\frac{x_1}{8}=-t$. Let $z_0=-t+ai$ then

$$(x-8t)\prod_{i=1}^{8}(x-z_i)=(x-8t)\left((x+t)^2+a^2\right)^4=x^9+3x^6c_2+3x^3c_2^2-x+c_2^3+c_1.$$

The last equality is not true for any c_1 and c_2. The lemma is proved.

Definition 1. It is known in [2], the determinant

$$R(f,g)=\begin{vmatrix} a_0 & a_1 & a_2 & \dots & a_n & 0 & \dots & 0\\ 0 & a_0 & a_1 & a_2 & \dots & a_n & \dots & 0\\ . & . & . & . & . & . & . & .\\ 0 & 0 & \dots & a_0 & a_1 & a_2 & \dots & a_n\\ b_0 & b_1 & b_2 & \dots & b_m & 0 & \dots & 0\\ 0 & b_0 & b_1 & b_2 & \dots & b_m & \dots & 0\\ . & . & . & . & . & . & . & .\\ 0 & 0 & \dots & b_0 & b_1 & b_2 & \dots & b_m \end{vmatrix}$$

is called ***resultant*** of the $f(x)$ and $g(x)$`.

From [2] discriminant of polynomial $f(x)$

$$D(f)=\prod_{i>j}^{9}\left(z_i-z_j\right)^2$$

is exactly equal to the resultant of the polynomials of $f(x)$ and $f'(x)$, i.e.

$$D(f)=\pm R(f,f').$$

We calculate the resultant

$$R(f,f')=$$

$$=\begin{vmatrix}
1&0&0&3c_2&0&0&3c_2^2&0&-1&c_2^3+c_1&0&0&0&0&0&0&0\\
0&1&0&0&3c_2&0&0&3c_2^2&0&-1&c_2^3+c_1&0&0&0&0&0&0\\
0&0&1&0&0&3c_2&0&0&3c_2^2&0&-1&c_2^3+c_1&0&0&0&0&0\\
0&0&0&1&0&0&3c_2&0&0&3c_2^2&0&-1&c_2^3+c_1&0&0&0&0\\
0&0&0&0&1&0&0&3c_2&0&0&3c_2^2&0&-1&c_2^3+c_1&0&0&0\\
0&0&0&0&0&1&0&0&3c_2&0&0&3c_2^2&0&-1&c_2^3+c_1&0&0\\
0&0&0&0&0&0&1&0&0&3c_2&0&0&3c_2^2&0&-1&c_2^3+c_1&0\\
0&0&0&0&0&0&0&1&0&0&3c_2&0&0&3c_2^2&0&-1&c_2^3+c_1\\
9&0&0&18c_2&0&0&9c_2^2&0&-1&0&0&0&0&0&0&0&0\\
0&9&0&0&18c_2&0&0&9c_2^2&0&-1&0&0&0&0&0&0&0\\
0&0&9&0&0&18c_2&0&0&9c_2^2&0&-1&0&0&0&0&0&0\\
0&0&0&9&0&0&18c_2&0&0&9c_2^2&0&-1&0&0&0&0&0\\
0&0&0&0&9&0&0&18c_2&0&0&9c_2^2&0&-1&0&0&0&0\\
0&0&0&0&0&9&0&0&18c_2&0&0&9c_2^2&0&-1&0&0&0\\
0&0&0&0&0&0&9&0&0&18c_2&0&0&9c_2^2&0&-1&0&0\\
0&0&0&0&0&0&0&9&0&0&18c_2&0&0&9c_2^2&0&-1&0\\
0&0&0&0&0&0&0&0&9&0&0&18c_2&0&0&9c_2^2&0&-1
\end{vmatrix}=$$

$$=\left(19683c_1^4+19683c_2^4+19683c_1^3c_2^3-4374c_1^2c_2^2+20736c_1c_2+4096\right)\cdot\left(19683c_1^4+19683c_2^4+19683c_1^3c_2^3+4374c_1^2c_2^2+20736c_1c_2-4096\right)$$

$$D(f)=R(f,f')=\left(19683c_1^4+19683c_2^4+19683c_1^3c_2^3-4374c_1^2c_2^2+20736c_1c_2+4096\right)\cdot$$
$$\cdot\left(19683c_1^4+19683c_2^4+19683c_1^3c_2^3+4374c_1^2c_2^2+20736c_1c_2-4096\right)$$

According to the lemma 1, the polynomial $f(x)$ does not havefour multiple complex roots. Therefore, the equation $D(f)=0$ defines the multiple real roots of the polynomial $f(x)$. Hence, if $D(f)=0$, then cubic parabolas

$$x=y^3+c_1$$
$$y=x^3+c_2$$

have a common tangent point, i.e. multiple rootsof $f(x)$. The equation $D(f)=0$ is equivalent to

$$9c_1^4+9c_2^4+9c_1^3c_2^3-2c_1^2c_2^2+\frac{256}{27}c_1c_2+\frac{4096}{2187}=0 \qquad (4)$$

and

$$9c_1^4+9c_2^4+9c_1^3c_2^3+2c_1^2c_2^2+\frac{256}{27}c_1c_2-\frac{4096}{2187}=0 \tag{5}$$

which will be considered as a function $c_2(c_1)$ given implicitly. How many ordinary functions are defined by implicit functions (4) and (5)?

a) Firstwe investigate the equation**(4).**Wecalculate the discriminant of polynomial (4)with respect to the variable c_2. By the known [2] formulas we find

$$D=-\left(c_1^2-\frac{16}{27}\right)^3\left(c_1^2+\frac{16}{27}\right)^3\left(c_1^4+\frac{1024}{729}\right)^2.$$

Since the equation (4) for D =0 has one real, for D< 0 tworeal roots and for D> 0 has no real root then we get the following statement.

Lemma 2. If $c_1<-\frac{4}{3\sqrt{3}}$ *or* $c_1>\frac{4}{3\sqrt{3}}$ then (4) defines two functions, they connected by the points $c_1=-\frac{4}{3\sqrt{3}}$ and $c_1=\frac{4}{3\sqrt{3}}$. And for $c_1>-\frac{4}{3\sqrt{3}}$ *and* $c_1<\frac{4}{3\sqrt{3}}$ the equation (4) does not define any real functions.

Lemma 3. The algebraic curve (4) splits the parameter (c_1,c_2) plane into three areas.

Proof. We express the symmetric function

$$\frac{D(f)}{19683}=c_1^4+c_2^4+c_1^3c_2^3-\frac{2}{9}c_1^2c_2^2+\frac{256}{243}c_1c_2+\frac{4096}{19683}$$

by the elementary symmetric functions $s=c_1+c_2,\ t=c_1c_2$ and get

$$\frac{D(f)}{19683}=\left((c_1+c_2)^2-2c_1c_2\right)^2-2c_1^2c_2^2+c_1^3c_2^3-\frac{2}{9}c_1^2c_2^2+\frac{256}{243}c_1c_2+\frac{4096}{19683}=$$
$$=\left(s^2-2t\right)^2+t^3-\frac{20}{9}t^2+\frac{256}{243}t+\frac{4096}{19683}=s^4-4ts^2+t^3+\frac{16}{9}t^2+\frac{256}{243}t+\frac{4096}{19683}$$

If $t<-\frac{4}{27}$, then the polynomial $D(f)$ is factorized

$$\frac{D(f)}{19683}=\left(s^2-\frac{4t+2\left|t-\frac{32}{27}\right|\sqrt{-t-\frac{4}{27}}}{2}\right)\left(s^2-\frac{4t-2\left|t-\frac{32}{27}\right|\sqrt{-t-\frac{4}{27}}}{2}\right).$$

Thus, the equation $D(f)=0$ splits into two equations

$$s^2-2t-\left|t-\frac{32}{27}\right|\sqrt{-t-\frac{4}{27}}=0 \qquad (6)$$

$$s^2-2t+\left|t-\frac{32}{27}\right|\sqrt{-t-\frac{4}{27}}=0 \qquad (7)$$

It remains to note that for $t>-\frac{4}{27}$ the discriminant is $D(f)\neq 0$.

Finally, the case

$$\begin{cases} D(f)=0 \\ t=-\frac{4}{27} \end{cases}$$

is also impossible, since $s^2=-\frac{8}{27}$, $t=-\frac{4}{27}$ then and the system of equations

$$\begin{cases} (c_1+c_2)^2=-\frac{8}{27} \\ c_1c_2=-\frac{4}{27} \end{cases}$$

has no real solutions.

Returning to the variables c_1 and c_2 we obtain two implicitly defined functions

$$c_1^2+c_2^2-\left|c_1c_2-\frac{32}{27}\right|\sqrt{-c_1c_2-\frac{4}{27}}=0 \qquad (8)$$

$$c_1^2+c_2^2+\left|c_1c_2-\frac{32}{27}\right|\sqrt{-c_1c_2-\frac{4}{27}}=0 \qquad (9)$$

Since, $-t-\frac{4}{27}=-c_1c_2-\frac{4}{27}>0$ the equation (8) defined two curves connected by the points $\left(-\frac{4}{3\sqrt{3}},\frac{4}{3\sqrt{3}}\right)$ and $\left(\frac{4}{3\sqrt{3}},-\frac{4}{3\sqrt{3}}\right)$. Considering the symmetry of (8) by c_1 and c_2, from lemma 2, we obtain that the two functions defined by (8) the graphsof them are on the set

$$\left\{(c_1,c_2):c_1\le-\frac{4}{3\sqrt{3}},c_2\le-\frac{4}{3\sqrt{3}}\right\}\cup\left\{(c_1,c_2):c_1\ge-\frac{4}{3\sqrt{3}},c_2\ge-\frac{4}{3\sqrt{3}}\right\}$$

And equation (9) with condition $-t-\frac{4}{27}=-c_1c_2-\frac{4}{27}>0$ doesnot define any curves.

It is clear that with $\left\{(c_1,c_2):c_1\le-\frac{4}{3\sqrt{3}},c_2\le-\frac{4}{3\sqrt{3}}\right\}\cup\left\{(c_1,c_2):c_1\ge-\frac{4}{3\sqrt{3}},c_2\ge-\frac{4}{3\sqrt{3}}\right\}$ we have

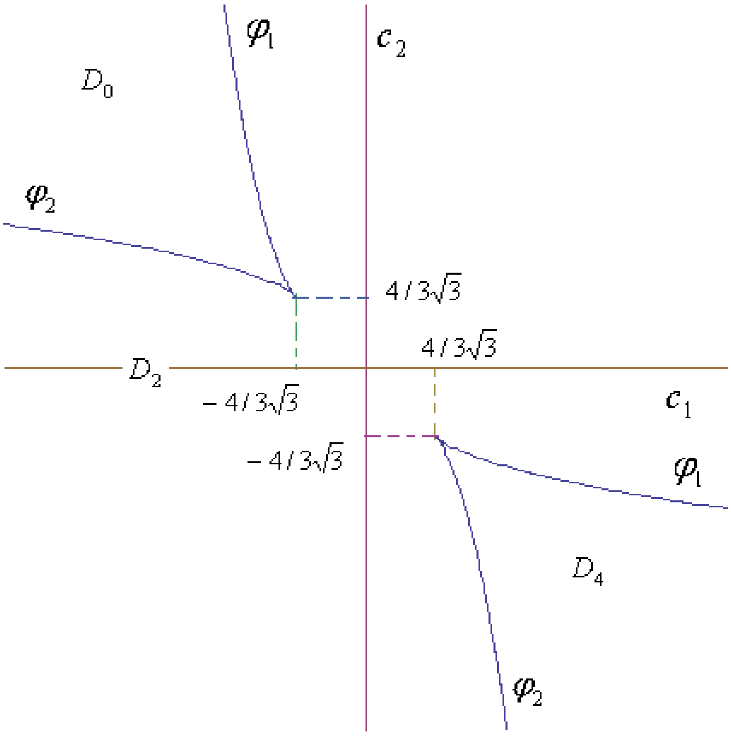
$$c_1^2+c_2^2-\left|c_1c_2-\frac{32}{27}\right|\sqrt{-c_1c_2-\frac{4}{27}}=0$$

Consequently, implicit function (8) defines two functions $c_2=\varphi_1(c_1)$ and $c_2=\varphi_2(c_1)$ they are defined on $\left(-\infty,-\frac{4}{3\sqrt{3}}\right]\cup\left[\frac{4}{3\sqrt{3}},+\infty\right)$, differentiable and monotonically decreasing. From symmetry (8) it follows that $\varphi_1^{-1}=\varphi_1$ and $\varphi_2^{-1}=\varphi_2$. Using the methods of implicit functions, we can prove that φ_1 and φ_2 are differentiable monotonically decreasing functions and they can be defined at the points $-\frac{4}{3\sqrt{3}}$ and $\frac{4}{3\sqrt{3}}$ by continuity

$$\varphi_1\left(-\frac{4}{3\sqrt{3}}\right)=\varphi_2\left(-\frac{4}{3\sqrt{3}}\right)=\frac{4}{3\sqrt{3}} \quad \text{and} \quad \varphi_1\left(\frac{4}{3\sqrt{3}}\right)=\varphi_2\left(\frac{4}{3\sqrt{3}}\right)=-\frac{4}{3\sqrt{3}}.$$

.

The graphs of these functions are depicted as follows.

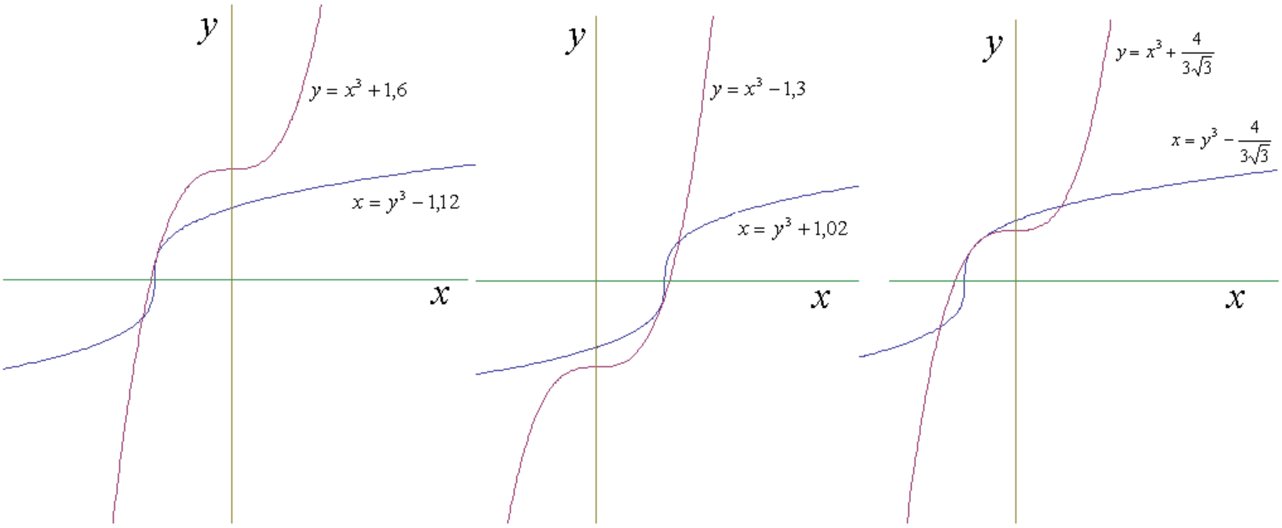


Picture 1.

So, the graphs of these functions split the (c_1,c_2) parameter plane into three open areas D_0, D_2 and D_4. If $(c_1,c_2)\in D_0$ or $(c_1,c_2)\in D_4$ then $f(x)$ have one real root, If $(c_1,c_2)\in D_2$ then $f(x)$mayhavemore real roots. If $(c_1,c_2)\in\varphi_1$ or $(c_1,c_2)\in\varphi_2$ then cubic parabolas

$$\begin{aligned} x&=y^3+c_1\\ y&=x^3+c_2 \end{aligned} \qquad (10)$$

are tangent(Pic 2.)



Picture 2.

b) Thenwe investigate the equation**(5).**We calculate the discriminant of polynomial (5) with respect to the variable c_2. By the known [2] formulas we find

$$D = -\left(c_1^2 - \frac{32}{27}\right)^2 \left(c_1^2 + \frac{32}{27}\right)^2 \left(c_1^4 + \frac{256}{729}\right)^3.$$

Since for any $c_1 \neq \pm\frac{4\sqrt{2}}{3\sqrt{3}}$ the discriminant of the equation (5) D< 0, it means that equation (5) has two real roots for any $c_1 \neq \pm\frac{4\sqrt{2}}{3\sqrt{3}}$.

Lemma 4. Equation (5) defines two functions for any c_1, and they intersect at the points $c_1 = -\frac{4\sqrt{2}}{3\sqrt{3}}$ and $c_1 = \frac{4\sqrt{2}}{3\sqrt{3}}$.

Lemma 5. The algebraic curve (5) splits the parameters (c_1, c_2) plane into five areas.

Proof. We express the symmetric function

$$\frac{D(f)}{19683} = c_1^4 + c_2^4 + c_1^3c_2^3 + \frac{2}{9}c_1^2c_2^2 + \frac{256}{243}c_1c_2 - \frac{4096}{19683}.$$

By the elementary symmetric functions $s = c_1 + c_2$, $t = c_1c_2$ and get

$$\frac{D(f)}{19683} = \left((c_1 + c_2)^2 - 2c_1c_2\right)^2 - 2c_1^2c_2^2 + c_1^3c_2^3 + \frac{2}{9}c_1^2c_2^2 + \frac{256}{243}c_1c_2 - \frac{4096}{19683} =$$

$$= \left(s^2 - 2t\right)^2 + t^3 - \frac{16}{9}t^2 + \frac{256}{243}t - \frac{4096}{19683} = s^4 - 4ts^2 + t^3 + \frac{20}{9}t^2 + \frac{256}{243}t - \frac{4096}{19683}$$

If $t<\frac{16}{27}$, then the polynomial $D(f)$ is factorized

$$\frac{D(f)}{19683}=\left(s^2-\frac{4t+2\sqrt{-\left(t-\frac{16}{27}\right)^3}}{2}\right)\left(s^2-\frac{4t-2\sqrt{-\left(t-\frac{16}{27}\right)^3}}{2}\right).$$

Thus, the equation $D(f)=0$ splits into two equations

$$s^2-2t-\sqrt{-\left(t-\frac{16}{27}\right)^3}=0 \qquad (11)$$

$$s^2-2t+\sqrt{-\left(t-\frac{16}{27}\right)^3}=0 \qquad (12)$$

It remains to note that for $t>\frac{16}{27}$ the discriminant is $D(f)\neq 0$.

Finally, the case

$$\begin{cases} D(f)=0 \\ t=\frac{16}{27} \end{cases}$$

also impossible, since $s^2=\frac{32}{27}$, $t=\frac{16}{27}$ then and the system of equations

$$\begin{cases} (c_1+c_2)^2=\frac{32}{27} \\ c_1c_2=\frac{16}{27} \end{cases}$$

has no real solutions.

Returning to the variables c_1 and c_2 we obtain two implicitly defined functions

$$c_1^2+c_2^2-\sqrt{-\left(c_1c_2-\frac{16}{27}\right)^3}=0 \qquad (13)$$

$$c_1^2+c_2^2+\sqrt{-\left(c_1c_2-\frac{16}{27}\right)^3}=0 \qquad (14)$$

Since, $-\left(t-\frac{16}{27}\right)=-c_1c_2+\frac{16}{27}>0$ the equation (13) defined two curves,they intersect at the point $\left(-\frac{4\sqrt{2}}{3\sqrt{3}},\frac{4\sqrt{2}}{3\sqrt{3}}\right)$ and $\left(\frac{4\sqrt{2}}{3\sqrt{3}},-\frac{4\sqrt{2}}{3\sqrt{3}}\right)$. Considering the symmetry of (13) by c_1 and c_2, from lemma 4, we obtain that the

two functions defined by (13) the graphs of them are on the set $(c_1, c_2) \in R^2$.
And equation (14) with condition $-\left(t-\frac{16}{27}\right)=-c_1c_2+\frac{16}{27}>0$ does not define any curves.
It is clear that with $(c_1, c_2) \in R^2$ we have

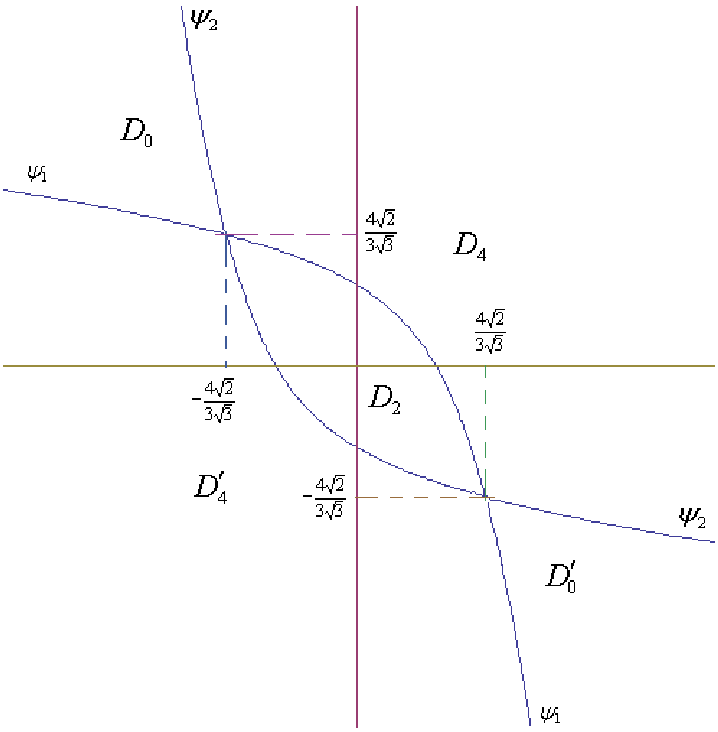
$$c_1^2+c_2^2-\sqrt{-\left(c_1c_2-\frac{16}{27}\right)^3}=0$$

Consequently, implicit function (13) defines two functions $c_2=\psi_1(c_1)$ and $c_2=\psi_2(c_1)$ they are defined on $(-\infty, +\infty)$, differentiable and monotonically decreasing. From symmetry (13) it follows that $\psi_1^{-1}=\psi_1$ and $\psi_2^{-1}=\psi_2$. Using the methods of implicit functions, we can prove that ψ_1 and ψ_2 are differentiable and monotonically decreasing functions and they intersect at the points $\left(-\frac{4\sqrt{2}}{3\sqrt{3}}, \frac{4\sqrt{2}}{3\sqrt{3}}\right)$ and $\left(\frac{4\sqrt{2}}{3\sqrt{3}}, -\frac{4\sqrt{2}}{3\sqrt{3}}\right)$.

$$\varphi_1\left(-\frac{4\sqrt{2}}{3\sqrt{3}}\right)=\varphi_2\left(-\frac{4\sqrt{2}}{3\sqrt{3}}\right)=\frac{4\sqrt{2}}{3\sqrt{3}} \quad \text{and} \quad \varphi_1\left(\frac{4\sqrt{2}}{3\sqrt{3}}\right)=\varphi_2\left(\frac{4\sqrt{2}}{3\sqrt{3}}\right)=-\frac{4\sqrt{2}}{3\sqrt{3}}$$

.

The graphs of these functions are depicted as follows.

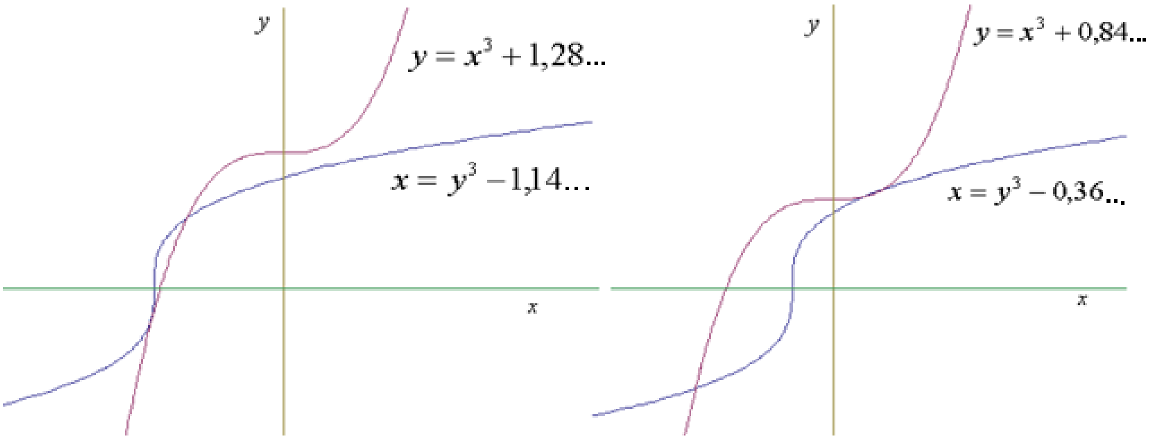


Picture 3.

So, the graphs of these functions split the (c_1, c_2) parameter plane into five open areas D_0, D_0', D_2, D_4 and D_4'. If (c_1, c_2) belongs to one of D_0, D_0', D_4, D_4' then $f(x)$ have one real root, If $(c_1, c_2) \in D_2$ then $f(x)$ can have more real roots. If $(c_1, c_2) \in \psi_1$ or $(c_1, c_2) \in \psi_2$ then cubic parabolas

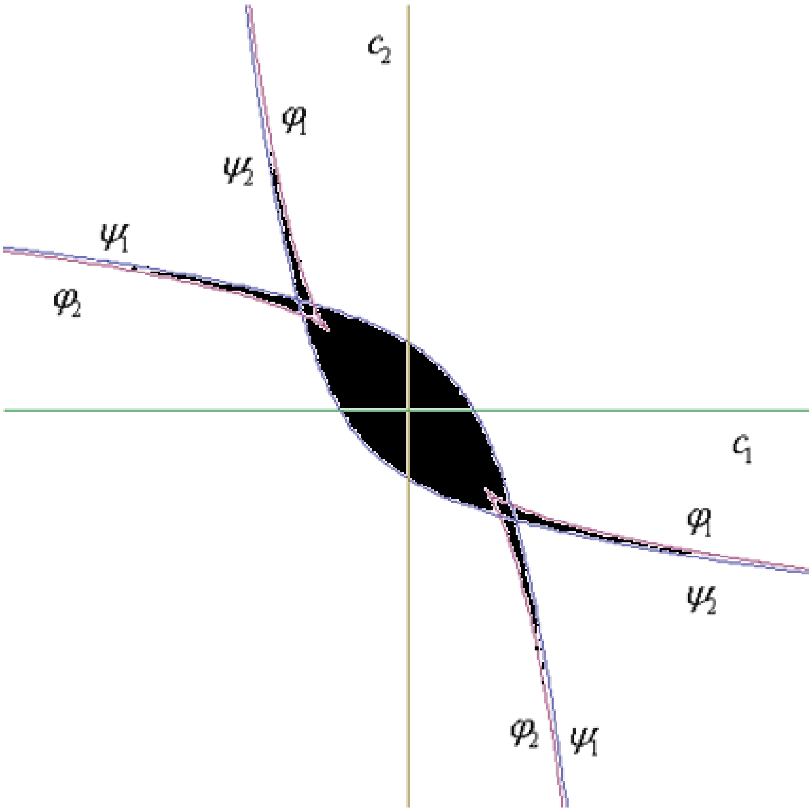
$$\begin{aligned} x &= y^3 + c_1 \\ y &= x^3 + c_2 \end{aligned} \tag{10a}$$

are tangent (Pic 4.)



Picture 4.

If we depict φ_1, φ_2, ψ_1 and ψ_2 on the (c_1, c_2) plane, we get the following picture.



Picture 5.

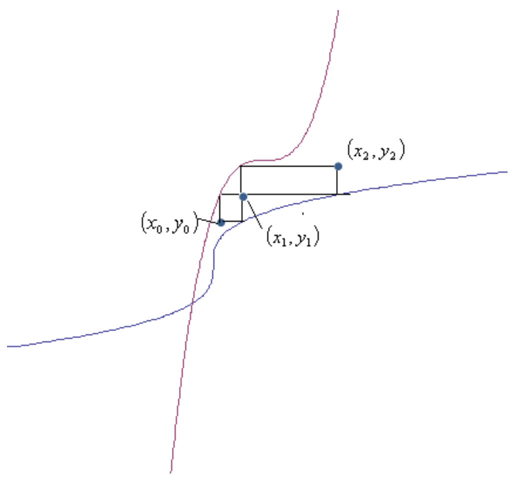
The black area is the Mandelbrot set which we will define on next part of this paper.

2. Graphical analysis.

In this part of the paper we introduce a geometric procedure that will help us understand the dynamics of some two-dimensional mappings. This procedure, called **graphical analysis**, enables us to use the graphs of functions to determine the behavior of orbits in many cases. Suppose we have the two-dimensional mapping

$$F_{c_1c_2}:\begin{cases} x' = f(y,c_1) \\ y' = g(x,c_2) \end{cases}$$

and wish to display the orbit of a given point (x_0, y_0). We begin by superimposing the graph of $x = f(y,c_1)$ on the graph of $y = g(x,c_2)$. The points of intersection of the graph $x = f(y,c_1)$ with the graph of $y = g(x,c_2)$ give us the **fixed points** of $F_{c_1c_2}$. To find the orbit of (x_0, y_0), we begin at the point (x_0, y_0) on the XOY plane. We first draw a horizontal line to the graph of $x = f(y,c_1)$. When this line meets the graph of $x = f(y,c_1)$, we have reached the point $(f(y_0,c_1), y_0)$ then draw a vertical line and denote it by V_1. We again begin at the point (x_0, y_0) on the XOY plane we draw a vertical line to the graph of $y = g(x,c_2)$. When this line meets the graph of $y = g(x,c_2)$, we have reached the point $(x_0, g(x_0,c_2))$ then draw a horizontal line and denote it by H_1. The intersection point of V_1 and H_1 is $(f(y_0,c_1), g(x_0,c_2)) = (x_1, y_1)$ the next point of the orbit of given point (x_0, y_0). To display the orbit of (x_0, y_0) geometrically, we thus continue this procedure over and over, in the next step we denote V_{i+1} instead of V_i and H_{i+1} instead of H_i. The intersection point of V_i and H_i is the i th point of the orbit of (x_0, y_0) by the mapping of $F_{c_1c_2}$. In the Picture6. we depicted graphical analysis of (x_0, y_0) by $\begin{cases} x_{n+1} = y_n^3 + c_1, \\ y_{n+1} = x_n^3 + c_2, \quad n = 0,1,2,... \end{cases}$



Picture 6.

For an arbitrary initial point (x_0, y_0) the orbits is determined by the following formula

$$x_{n+1} = y_n^3 + c_1$$
$$y_{n+1} = x_n^3 + c_2, \quad n = 0,1,2,...$$

It is tame that $F_{c_1c_2}$ mapping maps the vertical interval to the horizontal interval and inversely. Interval may be contract or extend. This proved the following theorem.

Theorem 2. If the mapping (2) has only one fixed point then every point except the fixed, tends to infinity, i. e. $x_n \to \infty,\ y_n \to \infty.$

Let $J(f)$ be the set of all points $(x_0, y_0) \in R^2$ that the orbit of them bounded.

Definition. The set $J(f)$ is called the filled Julia set.

Definition. The boundary of the filled Julia set is called Julia set.

By the graphical analysis we obtain the following theorems.

Theorem 3. If the mapping (2) has one fixed point then one $J(f)$ consist of only one point which is fixed.

Theorem4. $J(f)$is connected set if and only if the mapping (2) have fixed points more then one.

Theorem 5. Let the points $(p_1, q_1),(p_2, q_2),....,(p_n, q_n)$to be the fixed points for the mapping (2), the points $(p_i, q_i) \neq (p_j, q_j)$ are arbitrary two points of them, the points (p_i, q_j) and (p_j, q_i) are the periodic points with prime period two.

Proof. The points $(p_i, q_i) \neq (p_j, q_j)$ are fixed, let

$$\begin{cases} x_0 = p_i \\ y_0 = q_j \end{cases} \Rightarrow \begin{cases} x_1 = q_j^2 + c_1 = p_j \\ y_1 = p_i^2 + c_2 = q_i \end{cases} \Rightarrow \begin{cases} x_2 = q_i^2 + c_1 = p_i \\ y_2 = p_j^2 + c_2 = q_j. \end{cases}$$

We see $(p_i, q_j) \to (p_j, q_i) \to (p_i, q_j)$ by (2). The theorem is proved.

Statement 1.The equations for finding fixed points and for finding periodic points with period two are the same, therefore this theorem is true. In first section we learnt the properties of fixed points, many of them are true for the periodic points with period two.

Definition. The mapping F undergoes a ***saddle node bifurcation*** on the line (curve) $\xi(c_1, c_2) = 0$ depending on parameters defined on R^2:

1. For $\xi(c_1, c_2) < 0$ (resp. $\xi(c_1, c_2) > 0$), F has no fixed point on R^2.

2. For $\xi(c_1, c_2) = 0$, F has one fixed point on R^2 and this fixed point is neutral.

3. For $\xi(c_1,c_2)>0$ (resp. $\xi(c_1,c_2)<0$), F has two fixed points on R^2, one attracting one repelling.

Definition. The mapping F undergoes a ***period doubling bifurcation*** on the line (curve) $\xi(c_1,c_2)=0$ depending on parameters defined on R^2:

1. For $\xi(c_1,c_2)<0$ and $\xi(c_1,c_2)>0$, F has no cycles of the period 2 on $R^2$and there is unique fixed point $p_{c_1c_2}$ (may be more) for F on R^2.

2. For $\xi(c_1,c_2)<0$(resp. $\xi(c_1,c_2)>0$), F has no cycles of the period 2 on R^2 and $p_{c_1c_2}$ is attracting (resp. repelling).

3. For $\xi(c_1,c_2)=0$, $p_{c_1c_2}$ is neutral.

4. For $\xi(c_1,c_2)>0$ (resp. $\xi(c_1,c_2)<0$), there is unique 2-cycle $q^1_{c_1c_2}, q^2_{c_1c_2}$ on R^2 with $F(q^1_{c_1c_2})=q^2_{c_1c_2}$. This 2-cycle is attracting (resp. repelling). Meanwhile, $p_{c_1c_2}$ is repelling (resp. attracting).

5. As $\xi(c_1,c_2)\to 0+0$(resp. $\xi(c_1,c_2)\to 0-0$), this 2-cycle $q^i_{c_1c_2} \to p_{c_1c_2}$.

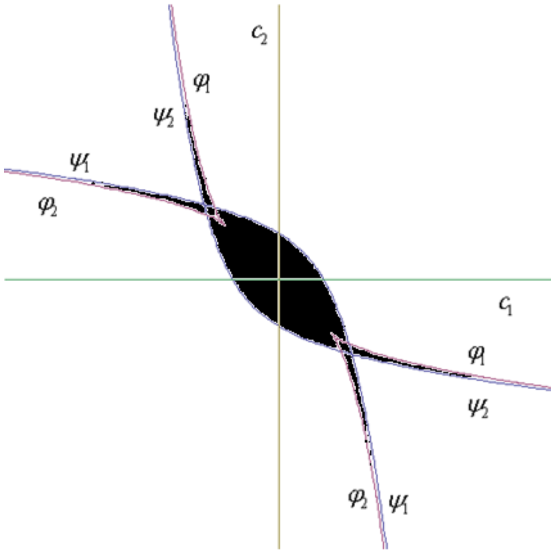
Definition. The lineswhich bifurcation occurs on them are called **bifurcation line** (=the analog of bifurcation point is in one dimension).

In our case the lines $\varphi_1,\varphi_2,\psi_1$ ***and*** ψ_2 ***are the bifurcation lines of the mapping (2).***

Definition: The **critical points** of the mapping F are all points (x_c, y_c) which determinant of Jacobian matrix at these points is equal to zero $\Delta(J(F(x_c,y_c)))=0$.

Definition: The **Mandelbrot set** $\mathrm{M_F}$ for the mapping F is the set of all points $(\mathrm{c_1},\mathrm{c_2})$on the parameter plane, which the orbits of the all critical pointsare bounded.

This picture is the Mandelbrot set for our mapping (2).



Picture 7.

REFERENCES

1. Sharkovsky, A. N. “Coexistence of Cycles of a Continuous Map of a Line into itself.” // Ukrain. Mat. Z 16 (1964), 61-71 (In Russian).

2. B. L. Van-der-Varden. Algebra. // M.: Mir, 1976. p-648

3. Devaney, R. L. A first course in chaotic dynamical systems. New York: Westview Press, 1992. p-321.

4. Devaney, R. L. An Introduction to Chaotic Dynamical Systems. New York: Westview Press, 1989. p-181.

5. Fatou, P. Sur les Equationes Fonctionelles. // Bull. Soc. Math. France 48 (1920), 33-94, 208-314.

6. Feigenbaum, M.J. Qualitative Universality for a Class of Nonlinear Transformations. //J. Stat. Phys. 19 (1978), 25-52.

7. Julia, G. Memoire sur l’Iteration des Fonctions Rationelles // J. Math Pures Appl. 4 (1918). 47-245.

8. Li, T., and Yorke, J. Period Three imply Chaos //American Mathematical Monthly 82 (1975), 985-992.

9. Mandelbrot, B. Fractals Aspects of the Iteration of $z \to \lambda z(1-z)$. Nonlinear dynamics, Annals of the New York Academy of science 357 (1980), 249-252.

10. Richard, A.H. A first Course in Discrete Dynamical Systems // Springer-Verlag. 1994. P-225.