NUMERICAL APPROXIMATION OF PERIODIC POINTS FOR SOME QUADRATIC MAPPINGS

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The logistic map is

$$x_{n+1} = rx_n(1 - x_n)$$

where x_n is a number between 0 and 1, the parameter r are those in the interval $[0,4]$ mean the condition for living in the island. After linear transformations we can consider the following as logistic mapping:

$$x_{n+1} = x_n^2 + c \tag{1}$$

but here, the parameter c changes between $[-2;0.25]$ and the number of population $|x_n|$. The learning of the asymptotics of trajectories of the mapping (1) is called the problem of Von Neumann - Ulam.

Two dimensional case of the mapping (1) is

$$F_{c_1c_2}: \begin{cases} x' = y^2 + c_1, \\ y' = x^2 + c_2. \end{cases} \tag{2}$$

where $(x, y) \in R^2$ and $(c_1, c_2) \in R^2$.

Our mathematical model of the population in the connected two islands is

$$F_{c_1c_2}: \begin{cases} x_{n+1} = y_n^2 + c_1, \\ y_{n+1} = x_n^2 + c_2. \end{cases} \tag{3}$$

where $|x_0|$ is the initial number of the population of first island and $|y_0|$ is the initial number of the population of second island in millions. For example, $|x_0| = 0.02$ means the initial number of population of first island is 20000. c_1 and c_2 are the living conditions in the islands respectively. $|x_n|$ and $|y_n|$ are the numbers of $n-$ th generation of populations first and second island.

To find the points where the period of mapping (2) is equal to four, it is necessary to solve the following equations

$$\begin{cases} \left(\left(\left(x^2+c_2\right)^2+c_1\right)^2+c_2\right)^2+c_1-x=0, \\ \left(\left(\left(y^2+c_1\right)^2+c_2\right)^2+c_1\right)^2+c_2-y=0. \end{cases}$$

Among the solutions of this system of equations are also points whose periods are equal to two. To separate them and leave only the equation of four points of period, we must divide the equations in the system of equations into the following two equations accordingly

$$x^4+2c_2x^2-x+c_2^2+c_1 \text{ ва } y^4+2c_1y^2-y+c_1^2+c_2.$$

In this case, the following system of equations is formed

$$\begin{cases} x^{12}+6c_2x^{10}+x^9+(15c_2^2+3c_1)x^8+4c_2x^7+(20c_2^3+12c_1c_2+1)x^6+ \\ +(2c_1+6c_2^2)x^5+(3c_1^2+4c_2+18c_1c_2^2+15c_2^4)x^4+(1+4c_1c_2+4c_2^3)x^3+ \\ +(c_1+6c_1^2c_2+5c_2^2+12c_1c_2^3+6c_2^5)x^2+(c_1^2+2c_2+2c_1c_2^2+c_2^4)x+ \\ +c_1^6+3c_1c_2^4+2c_2^3+3c_1^2c_2^2+2c_1c_2+c_1^3+1=0, \\ \\ y^{12}+6c_1y^{10}+y^9+(15c_1^2+3c_2)y^8+4c_1y^7+(20c_1^3+12c_1c_2+1)y^6+ \\ +(2c_2+6c_1^2)y^5+(3c_2^2+4c_1+18c_2c_1^2+15c_1^4)y^4+(1+4c_1c_2+4c_1^3)y^3+ \\ +(c_2+6c_2^2c_1+5c_1^2+12c_2c_1^3+6c_1^5)y^2+(c_2^2+2c_1+2c_2c_1^2+c_1^4)y+ \\ +c_2^6+3c_2c_1^4+2c_1^3+3c_1^2c_2^2+2c_1c_2+c_2^3+1=0. \end{cases}$$

According to Abel's theorem, these equations cannot be solved analytically in the general case. Therefore, we solve it using approximate solution methods for certain values of the parameters.

For example $c_1=-0.98$ and $c_2=-0.02$ let's solve approximately.

$$\begin{cases} 0.099144+0.919616\,x-1.09315\,x^2+1.07837\,x^3+2.79415\,x^4-1.9576\,x^5+ \\ +1.23504\,x^6-0.08\,x^7-2.934\,x^8+x^9-0.12\,x^{10}+x^{12}=0, \\ \\ -0.0115392-1.07565\,y-0.417991\,y^2-2.68637\,y^3+9.57098\,y^4+5.7224\,y^5- \\ -17.5886\,y^6-3.92\,y^7+14.346\,y^8+y^9-5.88\,y^{10}+y^{12}=0. \end{cases}$$

The equations in this system of equations are not related to each other so we solve them separately.

a. First

$$0.099144+0.919616x-1.09315x^2+1.07837x^3+2.79415x^4-1.9576x^5+$$
$$+1.23504x^6-0.08x^7-2.934x^8+x^9-0.12x^{10}+x^{12}=0$$

We solve numerical solutions of the equation using approximate methods. To do this, we find the gap where all the solutions are located.

$$A=\max\{|a_1|,|a_2|,\ldots,|a_n|\}=2.934,\quad R=1+\frac{A}{|a_0|}=1+\frac{2.934}{1}=3.934.$$

This means that all solutions are in the interval $(-3.934, 3.934)$.

$$f=0.099144+0.919616x-1.09315x^2+1.07837x^3+2.79415x^4-1.9576x^5+$$
$$+1.23504x^6-0.08x^7-2.934x^8+x^9-0.12x^{10}+x^{12},$$
$$f_1=0.919616-2.18631x+3.2351x^2+11.1766x^3-9.788x^4+7.41024x^5-$$
$$-0.56x^6-23.472x^7+9x^8-1.2x^9+12x^{11},$$
$$f_2=-0.099144-0.842981x+0.910962x^2-0.808776x^3-1.86276x^4+1.14193x^5-$$
$$-0.61752x^6+0.0333333x^7+0.978x^8-0.25x^9+0.02x^{10},$$
$$f_3=-744.5-6379.66x+6323.19x^2-5530.42x^3-14446.2x^4+7439.43x^5-$$
$$-3945.68x^6-97.04x^7+7346x^8-1287x^9,$$
$$f_4=0.020562+0.181177x-0.144408x^2+0.126777x^3+0.423907x^4-0.132206x^5+$$
$$+0.085444x^6+0.01774x^7-0.201121x^8,$$
$$f_5=5.0731-4.04273x+29.2359x^2+47.3173x^3+13.3759x^4+27.4597x^5+27.031x^6+5.8618x^7,$$
$$f_6=0.79745-1.00711x+4.99726x^2+6.49981x^3+0.109407x^4+4.10101x^5+3.33102x^6,$$
$$f_7=-0.329551-0.544621x-1.2825x^2+0.140093x^3-1.28697x^4-2.87271x^5,$$
$$f_8=-0.498182+1.88381x-3.20112x^2-5.13992x^3+0.896849x^4,$$
$$f_9=10.1898-35.1446x+58.6061x^2+111.845x^3,$$
$$f_{10}=-0.012912-0.0393259x-0.0202422x^2,$$
$$f_{11}=-111.409-201.795x$$
$$f_{12}=-0.00262955$$

	$f_0=f$	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	f_{10}	f_{11}	f_{12}	
-3.934	+	-	+	+	-	-	+	+	+	-	-	+	-	7
3.934	+	+	+	+	-	+	+	-	-	+	-	-	-	5

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We can see that there are two real solutions to the equation.

Algorithm:

1. We divide the interval $(-3.934, 3.934)$ into two equal parts and check which interval has a solution using the Sturm theorem for each of these intervals. If there is only one solution in each interval, we find approximate solutions using an arbitrary one of the methods of dividing the section into two equal parts, watts, and attempts to find solutions for each interval.

2. If more than one solution is in the same interval, then we apply the Sturm theorem again by dividing the interval into three equal parts. If several more solutions remain in the same interval, we will continue to use Sturm's theorem to divide the interval into four, five, six, and so on. We stop when there is only one solution or no solution in each interval.

3. Then we find the approximate solutions using the arbitrary one of the methods of dividing the section into two equal parts, watts and attempts, to separate the intervals in which there is a solution and find the solutions in each interval.

For the second equation, we use the same algorithm.

b. The second

$$-0.0115392-1.07565\,y-0.417991\,y^2-2.68637\,y^3+9.57098\,y^4+5.7224\,y^5-$$
$$-17.5886\,y^6-3.92\,y^7+14.346\,y^8+y^9-5.88\,y^{10}+y^{12}=0$$

We solve numerical solutions of the equation using approximate methods. To do this, we find the gap where all the solutions are located

$$A=\max\{|a_1|,|a_2|,\dots,|a_n|\}=17.5886,\quad R=1+\frac{A}{|a_0|}=1+\frac{17.5886}{1}=18.5886.$$

This means that all solutions are in the interval $(-18.5886, 18.5886)$.

$$g = -0.0115392 - 1.07565\,y - 0.417991\,y^2 - 2.68637\,y^3 + 9.57098\,y^4 + 5.7224\,y^5 -$$
$$-17.5886\,y^6 - 3.92\,y^7 + 14.346\,y^8 + y^9 - 5.88\,y^{10} + y^{12},$$
$$g_1 = -1.07565 - 0.835981\,y - 8.0591\,y^2 + 38.2839\,y^3 + 28.612\,y^4 - 105.532\,y^5 -$$
$$-27.44\,y^6 + 114.768\,y^7 + 9\,y^8 - 58.8\,y9 + 12\,y^{11},$$
$$g_2 = 0.0115392 + 0.986011\,y + 0.348326\,y^2 + 2.01478\,y^3 - 6.38065\,y^4 - 3.33807\,y^5 +$$
$$+8.79432\,y^6 + 1.63333\,y^7 - 4.782\,y^8 - 0.25\,y^9 + 0.98\,y^{10},$$
$$g_3 = 1.11169 + 4.05728\,y + 21.2208\,y^2 - 27.7251\,y^3 - 23.8725\,y^4 + 16.9743\,y^5 +$$
$$+14.0365\,y^6 - 1.98041\,y^7 - 3.93753\,y^8 - 0.536027\,y^9,$$
$$g_4 = 15.437 + 53.3631\,y + 287.127\,y^2 - 426.092\,y^3 - 274.673\,y^4 + 282.865\,y^5 +$$
$$+155.229\,y^6 - 54.8164\,y^7 - 46.3148\,y^8,$$
$$g_5 = -0.0107472 - 0.0728328\,y - 0.125708\,y2 + 0.659926\,y^3 - 0.648162\,y^4 + 0.0203181\,y^5 +$$
$$+0.307983\,y^6 - 0.132465\,y^7,$$
$$g_6 = -28.6209 - 146.467\,y - 466.802\,y^2 + 1191.69\,y^3 - 289.711\,y^4 - 484.562\,y^5 + 229.686\,y^6,$$
$$g_7 = 0.00719286 + 0.0711498\,y + 0.152208\,y^2 - 0.242717\,y^3 - 0.0750924\,y^4 + 0.0865886\,y^5\,,$$
$$g_8 = 4.9153 - 68.943\,y + 153.899\,y^2 + 11.987\,y^3 - 106.64\,y^4,$$
$$g_9 = -0.00418028 - 0.117396\,y - 0.00190424\,y2 + 0.125103\,y3,$$
$$g_{10} = -5.26161 + 62.781\,y - 53.9861\,y^2,$$
$$g_{11} = 0.0181738 - 0.037381\,y$$
$$g_{12} = -12.5005$$

	$g_0 = g$	g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8	g_9	g_{10}	g_{11}	g_{12}	
-18.5886	+	-	+	+	-	+	+	-	-	-	-	+	-	7
18.5886	+	+	+	-	-	-	+	+	-	+	-	-	-	5
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We can see that there are two real solutions to the equation.

As a result,

$$x_1 = -0.9798840171550919,\ \ x_2 = -0.09607531847629341$$
$$y_1 = -0.01076953317967882,\ \ y_2 = 0.9401726870760027.$$

This means that the four periods of a given mapping have four equal points.

$$(x_1, y_1) = (-0.9798840171550919,\ -0.01076953317967882),$$
$$(x_1, y_2) = (-0.9798840171550919, 0.9401726870760027),$$
$$(x_2, y_1) = (-0.09607531847629341,\ -0.01076953317967882),$$
$$(x_2, y_2) = (-0.09607531847629341, 0.9401726870760027).$$

At these points, we examine the spectra of the mapping given. That is, we find the modulus of the values of the equations in a given system of equations at points x and y, respectively. It follows that this period is attractive because the absolute values of multiplier smaller then one.

For our mapping (2).

$$F_{c_1c_2}:\begin{cases} x' = y^2 + c_1, \\ y' = x^2 + c_2. \end{cases} \quad (4)$$

If $c_1 = c_2 = -1$ then all point of out site the rectangle $|x| \le \frac{1}{2}(1+\sqrt{5})$, $|y| \le \frac{1}{2}(1+\sqrt{5})$ tend to infinity. Some inside points tend to fixed points $(-1,0)$ *or* $(0,-1)$. And some inside points tend to periodic points with period two $(0,0)$ *and* $(-1,-1)$.

For example $x_0 = 0.776$, $y_0 = -0.36$.

n	x_n	y_n
$n=1$	-0,8704	-0,397824
$n=2$	-0,841736065024	-0,24240384
$n=3$	-0,941240378353254	-0,291480396837912
...	...	...
$n=10$	-0,999970282135738	-3,57147346333631E-6
$n=11$	-0,999999999987245	-5,94348453725036E-5
$n=12$	0,999999996467499	-2,551092670E-11
$n=13$	-1	-7,0650016823E-9
$n=14$	-1	0

For (32) when $c_1 = c_2 = -1$ then filled Julia set Fig. 2.3.1.

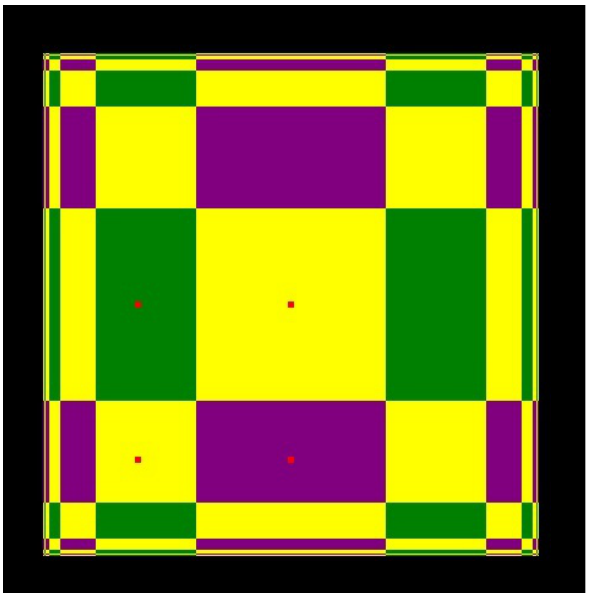

Figure 2.3.1. For $c_1 = c_2 = -1$ the classification of Julia set.

Let $c_1 = -0.98, c_2 = -0.02$ then all point in Julia set tend to the periodic points with period four.

For example $x_0 = 0.06, y_0 = -0.36.$

n	x_n	y_n
$n = 1$	-0,97973104	0,70318016
...	...	...
$n = 16$	0,0960753164810756	-0,0107895746542244
$n = 17$	-0,979883585078781	-0,0107695335630612
$n = 18$	-0,979884017146834	0,940171840306845
$n = 19$	-0,0960769106940412	0,940172687059817
$n = 20$	-0,0960753185067235	-0,0107692272314892

Let $c_1 = -1.22, c_2 = -0.38$ then all point in Julia set tend to the periodic points with period eight but there are two cyclical points with period eight.

Classification all Cauchy problems for $c_1 = -1.22, c_2 = -0.38$ on the Figure 2.3.2.

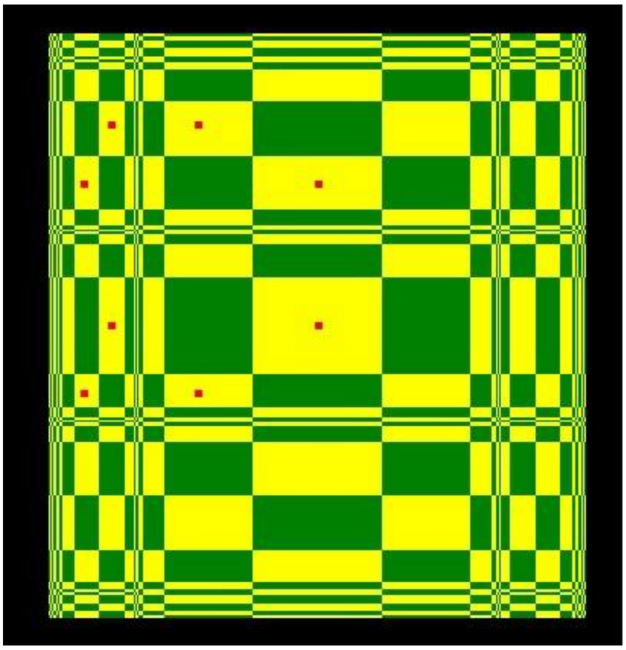

Figure 2.3.2. For $c_1 = -1.22, c_2 = -0.38$ the classification of Julia set.

Classification all Cauchy problems for $c_1 = -1.19, c_2 = -0.44.$ period 16, we get Fig. 2.3.3.

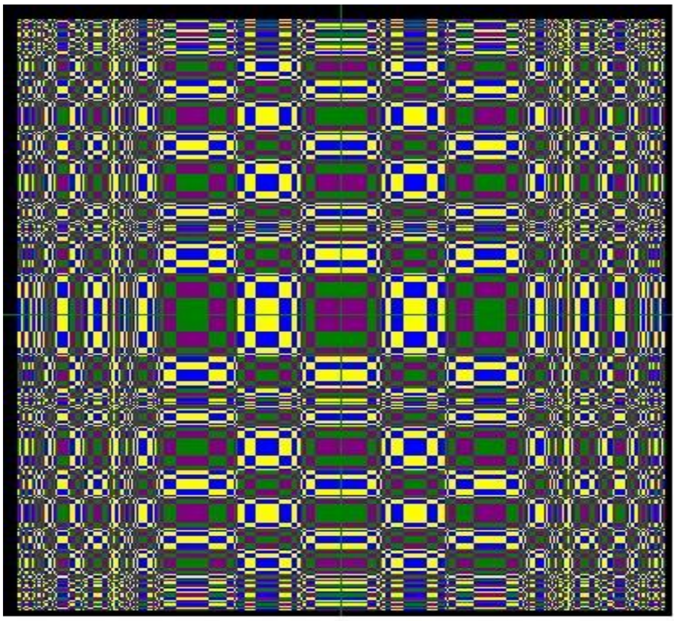

Figure 2.3.3.

Classification all Cauchy problems for $c_1 = -1.19, c_2 = -0.44$ Period 3 and 6.

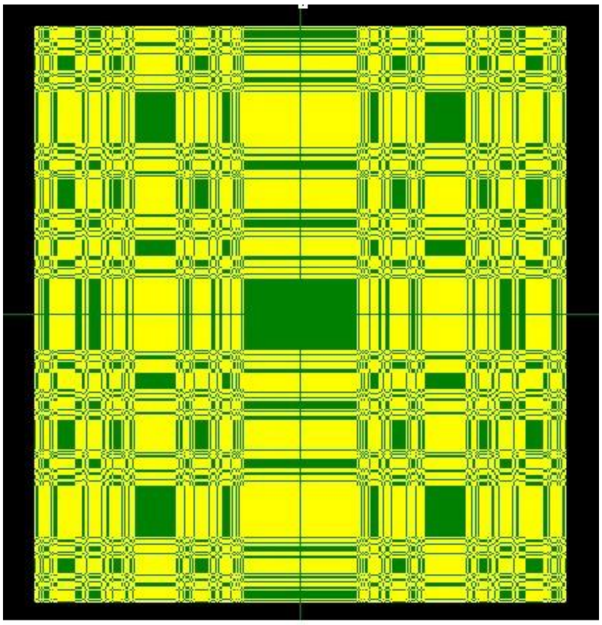

Figure 2.3.4. ***For*** $c_1 = -1.19, c_2 = -0.44$ ***the classification of Julia set.***

Classification all Cauchy problems for $c_1 = -1.31, c_2 = -0.8.$ Strange attractor.

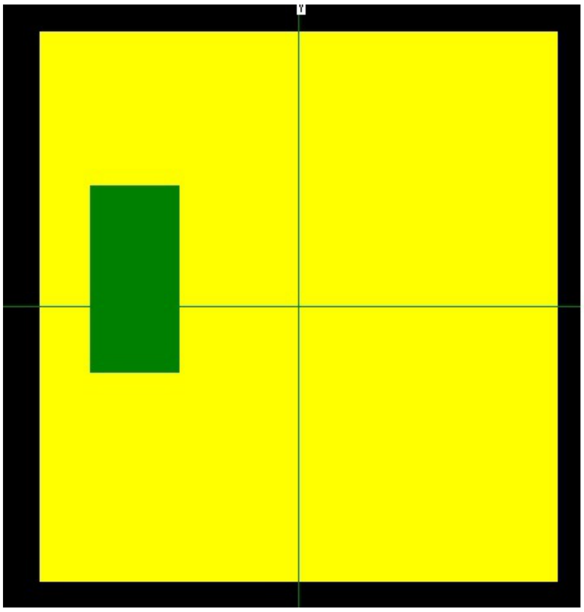

Figure 2.3.5. ***For*** $c_1 = -1.31, c_2 = -0.8.$ ***the classification of Julia set.***

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